

T-spectra and Poincaré Duality

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Abstract

Frank Adams introduced the notion of a complex oriented cohomology theory represented by a commutative ring-spectrum and proved the Poincaré Duality theorem for this general case. In the current paper we consider oriented cohomology theories on algebraic varieties represented by multiplicative symmetric *T*-spectra and prove the Duality theorem, which mimics the result of Adams. This result is held, in particular, for Motivic Cohomology and Algebraic Cobordism of Voevodsky.

0. Introduction

In certain cases a commutative ring-spectrum E can be equipped with a distinguished element $c \in E^2(\mathbb{P}^\infty)$ called a complex orientation of E (see [Ad]). The pair (E, c) is called a complex oriented ring spectrum. Given a complex orientation c of E , every smooth complex projective variety X can be equipped with a homological class $[X] \in E_{2d}(X)$ called the fundamental class of X (here d stays for the complex dimension of X). This class has the property that the cap-product

$$\frown[X] : E^*(X) \rightarrow E_{2d-*}(X)$$

provides an isomorphism of cohomology and homology groups of X . The isomorphism is often called the Poincaré Duality isomorphism.

From the modern point of view it looks pretty interesting to obtain an analogue of this result in the context of Algebraic Geometry. It is reasonable

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in this case to choose and fix a field k and consider a symmetric commutative ring T -spectrum \mathcal{A} in the sense of Voevodsky [Vo] (for the concept of symmetric T -spectrum see Jardine [Ja]). The T -spectrum \mathcal{A} determines bi-graded cohomology and homology theories ($A^{*,*}$ and $A_{*,*}$) on the category of algebraic varieties [Vo, p.595]. (We also assume the spectrum \mathcal{A} to be a ring-spectrum *i.e.* be endowed with a multiplication $\mu: \mathcal{A} \wedge \mathcal{A} \rightarrow \mathcal{A}$, which induces product structures in (co)homology.) In some cases \mathcal{A} can be equipped with a distinguished element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$, which Morel calls an orientation of \mathcal{A} . Following him the pair (\mathcal{A}, γ) is called an oriented symmetric commutative ring T -spectrum. The orientation γ equips both cohomology $A^{*,*}$ and homology $A_{*,*}$ with trace structures [PS2, Pi]. The latter means that for every projective morphism $f: Y \rightarrow X$ of k -smooth irreducible varieties with $d = \dim(X) - \dim(Y)$ there are two operators $f_! : A^{*,*}(Y) \rightarrow A^{*+2d, *+d}(X)$ and $f^! : A_{*,*}(X) \rightarrow A_{*-2d, *-d}(Y)$ satisfying a list of natural properties. Define now a fundamental class of a k -smooth projective equi-dimensional variety X of dimension d as $[X] \stackrel{\text{def}}{=} \pi^!(1) \in A_{2d, d}(X)$, where $\pi: X \rightarrow \text{pt}$ is the structure morphism. Our main result claims that the map

$$\frown [X]: A^{*,*}(X) \xrightarrow{\cong} A_{2d-*, d-*}(X)$$

is a grade-preserving isomorphism (Poincaré Duality isomorphism).

There are at least two interesting examples of oriented symmetric commutative ring T -spectra. The first one is a symmetric model MGL of the algebraic cobordism T -spectrum MGL of Voevodsky [Vo, p. 601]. This symmetric commutative ring T -spectrum MGL together with an orientation $\gamma \in \text{MGL}^{2,1}(\mathbb{P}^\infty)$ is described in details in [PY, Sect.6.5]. So that, every smooth irreducible projective variety X/k of dimension d has the fundamental class $[X] \in \text{MGL}_{2d, d}(X)$ and the cap-product with this class

$$\frown [X]: \text{MGL}^{*,*}(X) \xrightarrow{\cong} \text{MGL}_{2d-*, d-*}(X)$$

is an isomorphism.

The second example is the Eilenberg–Mac Lane T -spectrum H (it is intrinsically a symmetric T -spectrum representing the motivic cohomology). This T -spectrum H is constructed in [Vo, p.598] and we briefly describe its orientation here. Recall that for a smooth variety X/k the first Chern class of a line bundle with value in the motivic cohomology defines a functorial isomorphism $\text{Pic}(X) = \text{H}^{2,1}(X)$. Thus, $\mathbb{Z} = \text{H}^{2,1}(\mathbb{P}^\infty)$ and the class of the line bundle $\mathcal{O}(-1)$ over \mathbb{P}^∞ is a free generator of $\text{H}^{2,1}(\mathbb{P}^\infty)$. This class provides the required orientation of H . Similarly to the case of algebraic cobordism, one has the fundamental class $[X] \in \text{H}_{2d, d}^{\mathcal{M}}(X)$ in Motivic homology and the isomorphism:

$$\frown [X]: \text{H}_{\mathcal{M}}^{*,*}(X) \xrightarrow{\cong} \text{H}_{2d-*, d-*}^{\mathcal{M}}(X).$$

To embellish the latter result, let us mention that unlike the topological context in algebro-geometrical case the canonical pairing $H_{\mathcal{M}}^{*,*}(X) \otimes H_{*,*}^{\mathcal{M}}(X) \rightarrow H_{\mathcal{M}}^{*,*}(\text{pt})$ is generally degenerated even with rational coefficients [Vo2].

The paper is organized as follows. Section 1 is devoted to product structures in extraordinary cohomology and homology theories. In section 2 we formulate Poincaré Duality theorem and derive it from two projection formulae, which are proven in sections 3 and 4. Finally, in Appendices A and B we display some useful properties of orientable theories.

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Notation. Throughout the paper we use Greek letters to denote elements of cohomology groups and Latin for homological ones;

- Sm/k is a category of smooth quasi-projective algebraic varieties over a field k .
- Δ always denotes a diagonal morphism;
- Symbol $\mathbf{1}$ denotes trivial one-dimensional bundle;
- For a vector bundle \mathcal{E} over X we write $s(\mathcal{E})$ for its section sheaf;
- For a vector bundle \mathcal{E} over X we write \mathcal{E}^\vee for the dual to \mathcal{E} ;
- $\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}^*(s(\mathcal{E}^\vee)))$ is the projective bundle of lines in \mathcal{E} ;
- $0: =$ the point $[0: 0: \dots : 0: 1] \in \mathbb{P}^n$
- typically \mathbb{P}^n is regarded as a hyperplane in \mathbb{P}^{n+1}
- \mathbb{P}^∞ is a space defined in [Vo]
- $\text{pt} = \text{Spec } k$;

For the convenience of perception we usually move indexes up and down oppositely to the predefined positions of $*$ or $!$.

1. Some products in (co)homology

Consider a symmetric T -spectrum \mathcal{A} [Ja, p.505] endowed with a multiplication $\mu: \mathcal{A} \wedge \mathcal{A} \rightarrow \mathcal{A}$ making \mathcal{A} a symmetric commutative ring T -spectrum. Then the spectrum \mathcal{A} determines bigraded cohomology and homology theories on the category of algebraic varieties [Vo, p.595]. A ring structure in cohomology is then given by the cup-product satisfying the following commutativity law. For $\alpha \in A^{p,q}$ and $\beta \in A^{p',q'}$, one has:

$$\alpha \smile \beta = (-1)^{pp'} \epsilon^{qq'} (\beta \smile \alpha), \quad (1.1)$$

where $\epsilon: A^{*,*} \rightarrow A^{*,*}$ is the involution described in Appendix B.

Suppose that \mathcal{A} is endowed with an element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$ satisfying the following two conditions:

(i) $\gamma|_{\mathbb{P}^0} = 0 \in A^{2,1}(\mathbb{P}^0)$,

(ii) $\gamma|_{\mathbb{P}^1} = \Sigma_T(1) \in A_{\{0\}}^{2,1}(\mathbb{P}^1)$ is the T -suspension of the unit $1 \in A^{0,0}(\text{pt})$,

then the pair (\mathcal{A}, γ) is called *an oriented symmetric commutative T -spectrum*. If \mathcal{A} can be endowed with an element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$ satisfying the conditions (i) and (ii) then \mathcal{A} is called *an orientable symmetric commutative T -spectrum*. For such a T -spectrum $\epsilon = \text{id}$ by Lemma B.1 and the commutativity law is reduced to $\alpha \smile \beta = (-1)^{pp'} (\beta \smile \alpha)$. In this case it is convenient to set $A^0 = \bigoplus_{p,q} A^{2p,q}$, $A^1 = \bigoplus_{p,q} A^{2p-1,q}$, $A_0 = \bigoplus_{p,q} A_{2p,q}$, and $A_1 = \bigoplus_{p,q} A_{2p-1,q}$, where $A^{*,*}$ (resp. $A_{*,*}$) are (co)homology theories represented by the T -spectrum \mathcal{A} . The functors $A^* = A^0 \oplus A^1: Sm/k \rightarrow \mathbb{Z}/2\text{-Ab}$ and $A_* = A_0 \oplus A_1: Sm/k \rightarrow \mathbb{Z}/2\text{-Ab}$ are (co)homology theories taking values in the category of $\mathbb{Z}/2$ -graded abelian groups. Although all our duality results hold for bigraded (co)homology groups, we shall work, for simplicity, with the $\mathbb{Z}/2$ -grading just introduced.

Multiplicativity of the T -spectrum \mathcal{A} provides a canonical way [Sw, 13.50] to supply the functors A^* and A_* (contravariant and covariant, respectively) with a product structure consisting of two cross-products

$$\underline{\times}: A_p(X) \otimes A_q(Y) \rightarrow A_{p+q}(X \times Y), \quad \overline{\times}: A^p(X) \otimes A^q(Y) \rightarrow A^{p+q}(X \times Y)$$

and two slant-products

$$/: A^p(X \times Y) \otimes A_q(Y) \rightarrow A^{p-q}(X), \quad \backslash: A^p(X) \otimes A_q(X \times Y) \rightarrow A_{q-p}(Y).$$

One also defines two inner products

$$\smile: A^p(X) \otimes A^q(X) \rightarrow A^{p+q}(X), \quad \frown: A^p(X) \otimes A_q(X) \rightarrow A_{q-p}(X),$$

as $\alpha \smile \beta = \Delta^*(\alpha \overline{\times} \beta)$ and $\alpha \frown a = \alpha \backslash \Delta_*(a)$, correspondingly. The cup-product makes the group $A^*(X)$ an associative skew-commutative $\mathbb{Z}/2$ -graded

unitary ring and this structure is functorial. (Skew-commutativity is not obvious and implied by the orientability of \mathcal{A} as it is shown in Appendix B). The cap-product makes the group $A_*(X)$ a unital $A^*(X)$ -module ($1 \frown a = a$ for every $a \in A_*(X)$) and this structure is functorial in the sense that $\alpha \frown f_*(a) = f_*(f^*(\alpha) \frown a)$.

We shall need below the following associativity relations, which are completely analogous to ones existing in the topological context (see, for example, [Sw, 13.61]). For $\alpha \in A^*(X \times Y)$, $\beta \in A^*(Y)$, $\gamma \in A^*(X)$, $a \in A_*(Y)$, and $b \in A_*(X)$, we have:

A.1. $\alpha/(\beta \frown a) = (\alpha \smile p_Y^*(\beta))/a$

A.2. $\gamma \smile (\alpha/a) = (p_X^*(\gamma) \smile \alpha)/a$

A.3. $(\alpha/a) \frown b = p_*^X((\alpha \frown (a \times b)))$,

where p_X and p_Y denote the corresponding projections.

We shall also need the following functoriality property of the $/$ -product (comp. [Sw, 13.52.iii]). For morphisms $f: X \rightarrow X'$, $g: Y \rightarrow Y'$, and elements $\alpha \in A^*(X' \times Y')$ and $a \in A_*(Y)$, one has: $(f \times g)^*(\alpha)/a = f^*(\alpha/g_*(a))$.

For the final object pt in Sm/k one, clearly, has $A^*(\text{pt}) = A_*(\text{pt})$. This provides us with a distinguished element $[\text{pt}] \in A_0(\text{pt})$ (fundamental class of the point) such that for any smooth X and arbitrary $\alpha \in A^*(X)$, one has: $\alpha/[\text{pt}] = \alpha$. (Here we assume the standard identification $X \times \text{pt} = X$.) One can easily verify that the canonical isomorphism $A^*(\text{pt}) = A_*(\text{pt})$ may be written as $\alpha \mapsto \alpha \frown [\text{pt}]$. Throughout the paper we implicitly use this construction and usually denote $[\text{pt}]$ by 1.

2. Poincaré Duality Theorem

Let \mathcal{A} be an orientable symmetric commutative ring T -spectrum. Then the involution ϵ from (1.1) coincides with the identity as explained in Appendix B. So that, the commutativity law is reduced to $\alpha \smile \beta = (-1)^{pp'}(\beta \smile \alpha)$. Setting $A^0 = \bigoplus_{p,q} A^{2p,q}$, $A^1 = \bigoplus_{p,q} A^{2p-1,q}$, we see that the functor $A^* := A^0 \oplus A^1$ takes value in the category of skew-commutative $\mathbb{Z}/2$ -graded rings. For what follows it is convenient to give the following

DEFINITION 2.1. *Let \mathcal{A} be an orientable symmetric commutative ring T -spectrum. A Chern element is an element $\gamma \in A^0(\mathbb{P}^\infty)$ such that $\gamma|_{\mathbb{P}^0} = 0$ and the family $\{1, \gamma|_{\mathbb{P}^1}\} \subset A^0(\mathbb{P}^1)$ is a free basis of the free rank two $A^0(\text{pt})$ -module $A^0(\mathbb{P}^1)$. Another term, which can be used for a Chern element, is a non-homogeneous orientation of \mathcal{A} .*

A Chern element γ lifts to a Chern structure in the cohomology theory A^* in the sense of [PS3, Def.3.2] and to a commutative Chern structure in the

homology theory A_* (resp. [Pi, Def's.2.1.1, 2.2.12]). In fact, for every line bundle L over $X \in Sm/k$ there exists a diagram of the form $X \xleftarrow{p} X' \xrightarrow{f} \mathbb{P}(V)$, where V is a finite dimensional k -vector space, X' is a torsor under a vector bundle over X , and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic [PS3, 3.23]. Denote by $c(L)$ the class in $A^0(X)$ such that $p^*(c(L)) = f^*(\gamma|_{\mathbb{P}(V)})$. By [So] the element $c(L)$ is well-defined and the assignment $L \mapsto c(L)$ is a Chern structure in A^* . Moreover, the family of operators $c(L) \frown : A_*^Z(X) \rightarrow A_*^Z(X)$ forms a commutative Chern structure in the homology theory A_* .

Any Chern structure in A^* (resp. on A_*) lifts to a trace structure in the cohomology (resp. homology), see [PS2, Thm.4.1.2] (resp. [Pi, Thm.5.1.4]). Namely, to every projective morphism $f : Y \rightarrow X$ of smooth varieties over k one assigns two grade-preserving operators $f_! : A^*(Y) \rightarrow A^*(X)$ and $f^! : A_*(X) \rightarrow A_*(Y)$ satisfying a list of natural properties. Precise definitions of trace structures in a ring (co)homology theory is given in [PS2, Pi]. The operators $f_!$ and $f^!$ are called trace operators. (By historical reasons they called *integrations* in [PS2].) The trace structures $f \mapsto f_!$ and $f \mapsto f^!$ are explicit and unique up to the following normalization condition. For a smooth divisor $i : D \hookrightarrow X$:

$$i_! \circ i^* = i_!(1) \smile : A^*(X) \rightarrow A^*(X), \quad (2.1)$$

$$i_* \circ i^! = i_!(1) \frown : A_*(X) \rightarrow A_*(X), \quad (2.2)$$

and $i_!(1) = c(\mathcal{L}(D))$.

For a projective morphism $f : Y \rightarrow X$ the map $f_! : A^*(Y) \rightarrow A^*(X)$ is a two-side $A^*(X)$ -module homomorphism, *i.e.*

$$\begin{aligned} f_!(f^*(\alpha) \smile \beta) &= \alpha \smile f_!(\beta) \\ f_!(\alpha \smile f^*(\beta)) &= f_!(\alpha) \smile \beta. \end{aligned} \quad (2.3)$$

DEFINITION 2.2. *Let \mathcal{A} be an orientable symmetric commutative ring T -spectrum equipped with a Chern element $\gamma \in A^0(\mathbb{P}^\infty)$. For a smooth projective variety X with the structure morphism $\pi : X \rightarrow \text{pt}$ we call $\pi^!(1) \in A_0(X)$ the **fundamental class** of X in A_* and denote it by $[X]$.*

REMARK 2.3. *Definitely, the class $[X]$ depends rather on the pair (A_*, γ) than on the T -spectrum \mathcal{A} . However, we prefer to use this simplified notation, since we always keep in mind one chosen and fixed Chern element γ throughout the paper.*

With the notion of fundamental class in hands, one can define duality maps

$$\mathcal{D}^\bullet : A^*(X) \rightarrow A_*(X) \quad \text{as} \quad \mathcal{D}^\bullet(\alpha) = \alpha \frown [X] \quad (2.4)$$

and

$$\mathcal{D}_\bullet : A_*(X) \rightarrow A^*(X) \quad \text{as} \quad \mathcal{D}_\bullet(a) = \Delta_!(1)/a. \quad (2.5)$$

THEOREM 2.4 (Poincaré Duality). *Let \mathcal{A} be an orientable symmetric commutative ring T -spectrum equipped with a Chern element $\gamma \in A^0(\mathbb{P}^\infty)$. Then for every smooth projective variety X the maps \mathcal{D}^\bullet and \mathcal{D}_\bullet are mutually inverse isomorphisms.*

If $\gamma \in A^{2,1}$ and X is equi-dimensional of dimension d then $[X] \in A_{2d,d}(X)$. In this case the isomorphism \mathcal{D}^\bullet identifies $A^{p,q}$ with $A_{2d-p,d-q}$. One can extract the following nice consequence of the Poincaré Duality theorem, which enables us to interpret trace maps in a way topologists like to.

COROLLARY 2.5. *For projective varieties $X, Y \in Sm/k$ and a morphism $f : X \rightarrow Y$, one has:*

$$f_! = \mathcal{D}_\bullet^Y f_* \mathcal{D}_X^\bullet \quad \text{and} \quad f^! = \mathcal{D}_X^\bullet f^* \mathcal{D}_\bullet^Y,$$

where \mathcal{D}_X and \mathcal{D}_Y are introduced above duality operators for varieties X and Y , respectively.

Proof. To prove the first equality, one should just check that $f_* \mathcal{D}_X^\bullet = \mathcal{D}_Y^\bullet f_!$. Taking into account that $[X] = f^![Y]$, one immediately derives the desired relation from the First Projection Formula below (Theorem 2.6). The second statement can be proven in a similar way, but requires the “dual” projection formula that we do not consider here. \square

The proof of Theorem 2.4 is based on two projection formulae for cap- and slant-products.

THEOREM 2.6 (The First Projection Formula). *For $X, Y \in Sm/k$, a projective morphism $f : Y \rightarrow X$, and any elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$, the relation*

$$f_*(\alpha \frown f^!(a)) = f_!(\alpha) \frown a \quad (2.6)$$

holds in the group $A_*(X)$.

We need a few simple corollaries of this theorem.

COROLLARY 2.7. *Let $\tau : X \times X \rightarrow X \times X$ be the transposition morphism. Then for any elements $\alpha \in A^*(X)$, $\beta \in A^*(X \times X)$, and $a \in A_*(X \times X)$, we have:*

a).

$$\Delta_!(\alpha) \frown a = \Delta_!(\alpha) \frown \tau_*(a)$$

b).

$$\Delta_!(\alpha) \smile \beta = \Delta_!(\alpha) \smile \tau^*(\beta)$$

in $A_*(X \times X)$ ($A^*(X \times X)$, respectively).

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \text{id} \downarrow & & \downarrow \tau \\ X & \xrightarrow{\Delta} & X \times X. \end{array} \quad (2.7)$$

Since the map τ is flat, the square is transversal due to [Fu, B.7.4.]. By the base change property A.2, one has: $\Delta^! \circ \tau_* = \Delta^!$. By Theorem 2.6, one has:

$$\Delta_!(\alpha) \frown a = \Delta_*(\alpha \frown \Delta^!(a)) = \Delta_*(\alpha \frown \Delta^!(\tau_*(a))) = \Delta_!(\alpha) \frown \tau_*(a)$$

that implies **a).** To get **b)** one uses cohomological projection formula (2.3) instead. \square

THEOREM 2.8 (The Second Projection Formula). *Let $f : Y \rightarrow X$ be a projective morphism of smooth varieties. Let also $T \in \text{Sm}/k$. Then for every $\alpha \in A^*(T \times Y)$ and $a \in A_*(X)$, one has (in $A^*(T)$):*

$$\alpha / f^!(a) = F_!(\alpha) / a, \quad (2.8)$$

where $F = \text{id} \times f$.

COROLLARY 2.9. *Let X be a smooth projective variety. Then in $A^*(X)$, we have:*

$$\Delta_!(1) / [X] = 1. \quad (2.9)$$

Proof. Denote by $p : X \rightarrow \text{pt}$ the structure morphism and let $P = \text{id} \times p : X \times X \rightarrow X$ be the projection. By Theorem 2.8, one has:

$$\Delta_!(1) / [X] = \Delta_!(1) / p^!(1) = P_!(\Delta_!(1)) = 1. \quad (2.10)$$

\square

Now we derive the main result as an easy consequence of Corollaries 2.9 and 2.7.

Proof of Theorem 2.4. Let $p_1, p_2 : X \times X \rightarrow X$ denote corresponding projections. Observe that for every $\gamma \in A^*(X \times X)$ one has the relation $\Delta_!(1) \smile \gamma = \gamma \smile \Delta_!(1)$. (In fact, the element $\Delta_!(1)$ is of degree zero, because the map $\Delta_!(1)$ is grade-preserving.) Thus, one has:

$$\begin{aligned} \Delta_!(1) / (\alpha \frown [X]) &\stackrel{\mathbf{A.1}}{=} (\Delta_!(1) \smile p_2^*(\alpha)) / [X] \stackrel{2.7.\mathbf{b)}}{=} (\Delta_!(1) \smile p_1^*(\alpha)) / [X] \\ &= (p_1^*(\alpha) \smile \Delta_!(1)) / [X] \stackrel{\mathbf{A.2}}{=} \alpha \smile (\Delta_!(1) / [X]) = \alpha. \end{aligned} \quad (2.11)$$

On the other hand, using 2.7.a, one has:

$$\begin{aligned} (\Delta_!(1)/a) \cap [X] &\stackrel{\mathbf{A.3}}{=} p_*(\Delta_!(1) \cap (a \times [X])) = p_*(\Delta_!(1) \cap ([X] \times a)) \\ &\stackrel{\mathbf{A.3}}{=} (\Delta_!(1)/[X]) \cap a = a. \end{aligned} \quad (2.12)$$

□

To complete the prove of Theorem 2.4 one needs to check formulae 2.6 and 2.8.

3. Proof of The First Projection Formula.

It is convenient to introduce a class \mathfrak{V} of projective morphisms $f : Y \rightarrow X$ for which the relation

$$f_*(\alpha \cap f^!(a)) = f_!(\alpha) \cap a \quad (3.1)$$

holds in $A_*(X)$ for every elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$.

Obviously, this class is closed with respect to composition.

We prove Theorem 2.6 in several stages showing consequently that the following classes of morphisms are contained in the class \mathfrak{V} .

- Zero-section morphisms of line bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$;
- Closed embeddings $i : D \hookrightarrow X$ of smooth divisors;
- Zero-sections of a finite sum of line bundles:

$$s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n);$$

- Zero-sections of arbitrary vector bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$;
- Closed embeddings $i : Y \hookrightarrow X$;
- Projections $p : X \times \mathbb{P}^n \rightarrow X$;

LEMMA 3.1. *Let \mathcal{L} be a line bundle over a smooth variety Y . Then the zero-section $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$ belongs to \mathfrak{V} .*

Proof. The map s is a section of the projection map $p : \mathbb{P}(\mathbf{1} \oplus \mathcal{L}) \rightarrow Y$. Let $\alpha \in A^*(Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{L}))$. The desired relation follows from (2.2) and (2.1):

$$\begin{aligned} s_*(\alpha \cap s^!(a)) &= s_*(s^*p^*(\alpha) \cap s^!(a)) = p^*(\alpha) \cap s_*s^!(a) \\ &= p^*(\alpha) \cap (s_!(1) \cap a) = s_!(s^*p^*(\alpha)) \cap a = s_!(\alpha) \cap a. \end{aligned} \quad (3.2)$$

□

PROPOSITION 3.2. *Let $X, Y \in \text{Sm}/k$, $i: Y \hookrightarrow X$ be a closed embedding with a normal bundle \mathcal{N} . If the zero-section morphism $s: Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{N})$ belongs to \mathfrak{V} then i belongs to \mathfrak{V} .*

Proof. Consider the following deformation diagram, in which B is the blowup of $X \times \mathbb{A}^1$ at $Y \times \{0\}$. This diagram has transversal squares.

$$\begin{array}{ccccc}
 & & B - Y \times \mathbb{A}^1 & & (3.3) \\
 & & \downarrow k_B & & \\
 \mathbb{P}(\mathbf{1} \oplus \mathcal{N}) & \xrightarrow{k_0} & B & \xleftarrow{k_1} & X \\
 \uparrow s & \searrow p & \uparrow t & & \uparrow i \\
 Y & \xrightarrow{j_0} & Y \times \mathbb{A}^1 & \xleftarrow{j_1} & Y
 \end{array}$$

One can easily see that the left-hand part of our diagram satisfies the conditions of Lemma A.5.

First, we shall show that the morphism t in Diagram 3.3 belongs to the class \mathfrak{V} . Let $\alpha \in A^*(Y \times \mathbb{A}^1)$ and $a \in A_*(B)$. Using Lemma A.5 we can rewrite a as $k_*^B(a_B) + k_*^0(a_0)$, where $a_0 \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{N}))$ and $a_B \in A_*(B - Y \times \mathbb{A}^1)$. From the Gysin exact sequence, we have:

$$t^!k_*^B = 0 \quad \text{and} \quad (3.4)$$

$$k_B^*t_! = 0. \quad (3.5)$$

Therefore, $t_*(\alpha \cap t^!k_*^B(a_B)) = 0$ and $t_!(\alpha) \cap k_*^B(a_B) = 0$. (The second relation yields from 3.5: $t_!(\alpha) \cap k_*^B(a_B) = k_*^B(k_B^*t_!(\alpha) \cap a) = 0$.) Thus, one has:

$$t_*(\alpha \cap t^!(a)) = t_*(\alpha \cap t^!k_*^0(a_0)). \quad (3.6)$$

Applying Lemma A.3 to the left-hand-side square of Diagram 3.3 and denoting $j_0^*(\alpha)$ by α_0 , one has:

$$t_*(\alpha \cap t^!k_*^0(a_0)) = k_*^0s_* \left(\alpha_0 \cap s^!(a_0) \right). \quad (3.7)$$

Similarly:

$$t_!(\alpha) \cap a = k_*^0(s_!(\alpha_0) \cap a_0). \quad (3.8)$$

By the proposition assumption, we have the relation $s_*(\alpha_0 \cap s^!(a_0)) = s_!(\alpha_0) \cap a_0$. Combining this with equalities 3.6, 3.7, and 3.8, one gets:

$$t_*(\alpha \cap t^!(a)) = t_!(\alpha) \cap a. \quad (3.9)$$

We now move the desired relation one more step further to the right in Diagram 3.3 and show that $i \in \mathfrak{V}$. Observe that k_*^1 is a monomorphism. Therefore, it suffices to check that for every elements $\alpha_1 \in A^*(Y)$ and $a_1 \in A_*(X)$ we have:

$$k_*^1i_*(\alpha_1 \cap i^!(a_1)) = k_*^1(i_!(\alpha_1) \cap a_1). \quad (3.10)$$

Setting $\alpha = (j_1^*)^{-1}(\alpha_1) \in A^*(Y \times \mathbb{A}^1)$, $a = k_*^1(a_1) \in A_*(B)$, and applying Lemma A.3 to the right-hand-side square of Diagram 3.3, one has: $k_*^1 i_* (\alpha_1 \frown i^!(a_1)) = t_*(\alpha \frown t^!(a))$. In the same way: $k_*^1(i_!(\alpha_1) \frown a_1) = t_!(\alpha) \frown k_*^0(a_0) = t_!(\alpha) \frown a$. Combining these two relations with 3.9, one sees that $i \in \mathfrak{V}$. \square

COROLLARY 3.3. *For a smooth divisor $i: D \hookrightarrow X$ the morphism i lies in \mathfrak{V} .*

COROLLARY 3.4. *Let $\mathcal{W} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ be an n -dimensional vector bundle over a variety Y which splits in the sum of line bundles. Then the zero-section morphism $s: Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{W})$ belongs to the class \mathfrak{V} .*

Proof. Apply Corollary 3.3 to each step of the filtration

$$Y \xrightarrow{i_1} \mathbb{P}(\mathbf{1} \oplus \mathcal{L}_1) \xrightarrow{i_2} \cdots \xrightarrow{i_n} \mathbb{P}(\mathbf{1} \oplus \mathcal{W}), \quad (3.11)$$

where the morphisms i_j are zero-sections of \mathcal{L}_j . \square

In order to proceed with the case of an arbitrary vector-bundle, we need the homological analogue of the splitting principle. Consider a vector bundle $\mathcal{E} \rightarrow Y$ of constant rank n over a smooth irreducible variety Y . Let \mathcal{GL}_n be the corresponding principal GL_n -bundle over Y , $T_n \subset GL_n$ be the diagonal tori, and $Y' = \mathcal{GL}_n/T_n$ be the orbit variety with the projection morphism $p: Y' \rightarrow Y$. Finally, we denote by $\mathcal{E}' = \mathcal{E} \times_Y Y'$ the pull-back of the vector bundle \mathcal{E} .

PROPOSITION 3.5. *The bundle \mathcal{E}' splits in a direct sum of line bundles and the map $p_*: A_*(Y') \rightarrow A_*(Y)$ is a universal splitting epimorphism (i.e. for any base-change $Z \rightarrow Y$ the induced map $A_*(Z \times_Y Y') \rightarrow A_*(Z)$ is a splitting epimorphism).*

Proof. The projection $\mathcal{GL}_n \rightarrow Y'$ and the natural T_n -action on \mathcal{GL}_n makes it a principal T_n -bundle over Y' . Moreover, if $\mathcal{GL}'_n = \mathcal{GL}_n \times_Y Y'$ is the pull-back of \mathcal{GL}_n , there is a natural isomorphism of principal GL_n -bundles

$$\mathcal{GL}_n \times_{T_n} GL_n \rightarrow \mathcal{GL}'_n \quad (3.12)$$

over Y' . The bundle \mathcal{E}' over Y' corresponds exactly to the principal GL_n -bundle \mathcal{GL}'_n . Thus, the mentioned isomorphism of principal GL_n -bundles over Y' shows that the bundle \mathcal{E}' splits in a direct sum of line bundles (say corresponding to the fundamental characters $\chi_1, \chi_2, \dots, \chi_n$ of the tori T_n). This proves the first assertion of the proposition.

To prove the second one, consider a Borel subgroup B_n in GL_n (say the subgroup of all upper triangle matrices) and let U_n be the maximal unipotent subgroup of B_n (the group of upper triangle matrices with 1's on the diagonal). Let $\mathcal{F} = \mathcal{GL}_n/B_n$ (this is just the flag bundle over Y associated to \mathcal{E}). The

bundle \mathcal{F} comes equipped with projections $q : \mathcal{F} \rightarrow Y$ and $r : Y' \rightarrow \mathcal{F}$, where the projection r is induced by the inclusion $T_n \subset B_n$. Using the natural U_n -action on \mathcal{GL}_n , it is easy to check that there is a tower of morphisms:

$$\mathcal{GL}_n = S_m \rightarrow S_{m-1} \rightarrow \cdots \rightarrow S_1 = \mathcal{F}, \quad (3.13)$$

which has a principal \mathbb{G}_a -bundle on each level (each level is a torsor over the trivial rank one vector bundle). By the strong homotopy invariance property [PS3, 2.2.6], the induced map on homology $r_* : A_*(Y') \rightarrow A_*(\mathcal{F})$ is an isomorphism.

As it was already mentioned, \mathcal{F} is a full flag bundle over Y associated to the bundle \mathcal{E} . Thus, there is a tower of morphisms

$$\mathcal{F} = Z_s \rightarrow Z_{s-1} \rightarrow \cdots \rightarrow Z_1 = Y \quad (3.14)$$

in which each level is a projective bundle associated to a vector bundle. By the Projective Bundle Theorem (PBT) A.6, we have a split epimorphism in homology induced on each floor. Therefore, the map $q_* : A_*(\mathcal{F}) \rightarrow A_*(Y)$ is a split epimorphism as well.

These proves that the map $p_* : A_*(Y') \rightarrow A_*(Y)$ is also an epimorphism.

One can easily check that all necessary properties of the morphisms p, q , and r are base-change invariant. Therefore, the constructed splitting epimorphism is universal. \square

PROPOSITION 3.6. *Let $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$ be the zero-section of the finite-dimensional vector bundle \mathcal{V} . Then $s \in \mathfrak{Q}$.*

Proof. Letting Y' be as above, denote by \mathcal{V}' the pull-back of the bundle \mathcal{V} with respect to the morphism p . Then by Proposition 3.5 the bundle \mathcal{V}' splits in a direct sum of line bundles and the induced map

$$\bar{p}_* : A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V}')) \rightarrow A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V})) \quad (3.15)$$

is an epimorphism.

Let $s : Y \rightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$ and $\bar{s} : Y' \rightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V}')$ be morphisms induced by zero-sections of the corresponding vector bundles. Then the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathbf{1} \oplus \mathcal{V}') & \xrightarrow{\bar{p}} & \mathbb{P}(\mathbf{1} \oplus \mathcal{V}) \\ \bar{s} \uparrow & & \uparrow s \\ Y' & \xrightarrow{p} & Y \end{array} \quad (3.16)$$

is transversal.

Let $\alpha \in A^*(Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V}))$. Choosing $b \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V}'))$ such that $a = \bar{p}_*(b)$ and applying Lemma A.3, one gets:

$$s_*(\alpha \frown s^!(a)) = \bar{p}_* \bar{s}_*(p^*(\alpha) \frown \bar{s}^!(b)) \quad (3.17)$$

and

$$s_!(\alpha) \frown a = \bar{p}_*(\bar{s}_!p^*(\alpha) \frown b). \quad (3.18)$$

Two expressions on the right-hand-sides coincide by Proposition 3.4. \square

COROLLARY 3.7. *Let $i: Y \hookrightarrow X$ be a closed embedding. Then, $i \in \mathfrak{A}$.*

Proof. Applying Proposition 3.2 we reduce the question to the case of the zero-section morphism $s: Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{N})$ of the normal bundle $\mathcal{N} = \mathcal{N}_{X/Y}$. The morphism s belongs to \mathfrak{A} by Proposition 3.6. \square

In order to check that projection morphisms $p: X \times \mathbb{P}^* \rightarrow X$ belong to \mathfrak{A} we need a few auxiliary results (3.9–3.11).

NOTATION 3.8. *For a projective morphism f we denote, from now on, the map $f_*f^!$ by f^\diamond and $f_!f^*$ by f_\diamond .*

LEMMA 3.9. a). *(left distributivity) Let a, b, c , and p be projective morphisms. If $a^\diamond = b^\diamond + c^\diamond$ then $(pa)^\diamond = (pb)^\diamond + (pc)^\diamond$, provided that both sides of the equality are well defined.*

b). *Given a transversal square with projective morphisms f and g*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ F \downarrow & \searrow h & \downarrow f \\ X & \xrightarrow{g} & Z \end{array}$$

one has the following equalities: $h^\diamond = g^\diamond f^\diamond = f^\diamond g^\diamond$.

c). *In the square above: $g_*F^\diamond = f^\diamond g_*$.*

d). *Let s_i be the standard embedding $\mathbb{P}^{n-i} \hookrightarrow \mathbb{P}^n$ and $p_n: \mathbb{P}_X^n \rightarrow X$ be the projection map. Let ψ_i be the same as in the Projective Bundle Theorem (see A.6). Then $p_n^n s_i^\diamond = \psi_i$.*

Proof. Part **a)** immediately follows from the definition of the operation \diamond , **b)** and **c)** are trivial corollaries of the transversal base-change property, **d)** easily follows from the PBT. \square

Fix now a variety $X \in Sm/k$ and take the n -dimensional projective space \mathbb{P}_X^n over X . (Up to the end of this section all the schemes are considered over the base scheme X and the product is implicitly taken over X .) Due to the PBT, the element $\Delta_!(1) \in A^*(\mathbb{P}^n \times \mathbb{P}^n)$ may be decomposed as

$$\Delta_!(1) = 1 \boxtimes \zeta^n + \zeta^n \boxtimes 1 + \sum_{i,j=1}^n a_{ij} \zeta^i \boxtimes \zeta^j, \quad (3.19)$$

where $\zeta = e(\mathcal{O}(1))$ is the canonical generator of $A^*(\mathbb{P}^n)$ as an $A^*(X)$ -algebra and $a_{ij} \in A^*(X)$ (see [PS1, Lemma 1.9.3]).

This equality together with the previous lemma gives us the following decomposition of the identity operator $\text{id}_{\mathbb{P}^n}$. Taking into account the relation $s_{ij}^\diamond(x) = (\zeta^i \boxtimes \zeta^j) \frown x$, where $s_{ij}: \mathbb{P}^{n-i} \times \mathbb{P}^{n-j} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$ is the standard embedding, we can rewrite the cap-product with $\Delta_!(1)$ operator in the form:

$$\Delta^\diamond = (\Delta_!(1) \frown) = s_{0n}^\diamond + s_{n0}^\diamond + \sum_{i,j=1}^n a_{ij} s_{ij}^\diamond. \quad (3.20)$$

Applying Lemma 3.9 to transversal squares:

$$\begin{array}{ccc} \mathbb{P}^{n-i} \times \mathbb{P}^{n-j} & \xrightarrow{\quad} & \mathbb{P}^n \times \mathbb{P}^{n-j} \\ \downarrow & \searrow^{p_{1,n} s_{ij}} & \downarrow p_{1,n-j} \\ \mathbb{P}^{n-i} & \xrightarrow{s_i} & \mathbb{P}^n \end{array} \quad (3.21)$$

(where we denote by $p_{1,k}$ the projection map $\mathbb{P}^n \times \mathbb{P}^k \rightarrow \mathbb{P}^n$), one gets the following equality:

$$\text{id} = (p_{1,n} \Delta)^\diamond = \sum_{i,j=0}^n a_{ij} (p_{1,n} s_{ij})^\diamond = \sum_{i,j=0}^n a_{ij} p_{1,n-j}^\diamond s_i^\diamond. \quad (3.22)$$

(here we set $a_{i0} = a_{0i} = 0$, unless $i = n$.)

LEMMA 3.10. *For the projection morphism $p_n: \mathbb{P}^n \rightarrow X$, we have:*

- a). $p_n^\diamond = -\sum_{j=1}^n a_{nj} p_{n-j}^\diamond$;
- b). $p_\diamond^n = -\sum_{j=1}^n a_{nj} p_\diamond^{n-j}$.

Proof. Let us check the first statement. For $n = 0$ we, trivially, have $p_0^\diamond = \text{id}$. Applying the map p_*^n to both sides of 3.22 and then using Lemma 3.9.c for the transversal squares

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^{n-j} & \longrightarrow & \mathbb{P}^{n-j} \\ p_{1,n-j} \downarrow & & \downarrow p_{n-j} \\ \mathbb{P}^n & \xrightarrow{p_n} & X, \end{array} \quad (3.23)$$

one gets:

$$p_*^n = \sum_{i,j=0}^n a_{ij} p_*^n p_{1,n-j}^\diamond s_i^\diamond = \sum_{i,j=0}^n a_{ij} p_{n-j}^\diamond (p_n^* s_i^\diamond). \quad (3.24)$$

Taking into account that due to 3.9.d $p_*^n s_i^\diamond = \psi_i$, ($p_n^* = \psi_0$), one has:

$$\left(p_n^\diamond + \sum_{j=1}^n a_{nj} p_{n-j}^\diamond \right) \psi_n = - \sum_{i,j=1}^{n-1} (\dots) \psi_i. \quad (3.25)$$

By the PBT, for any $x \in A_*(X)$ we can choose an element $\varphi(x) \in A_*(\mathbb{P}_X^n)$ such that $\psi_i(\varphi(x)) = \begin{cases} 0, & i < n \\ x, & i = n. \end{cases}$

Applying operators on both sides of (3.25) to $\varphi(x)$, we get:

$$0 = p_n^\diamond + \sum_{j=1}^n a_{nj} p_{n-j}^\diamond \quad (3.26)$$

This finishes the proof of case a). The cohomological relation b) may be proved by dualization of these arguments or found in [PS1, Section 1.10]. \square

PROPOSITION 3.11. *Let p_n denote, as before, the projection morphism $p_n: \mathbb{P}_X^n \rightarrow X$. Then for every element $a \in A_*(X)$, one has:*

$$p_*^n(p_n^!(a)) = p_!^n(1) \frown a.$$

Proof. Rewriting the Proposition statement in our notation, we should verify the relation $p_n^\diamond(a) = p_n^n(1) \frown a$. We proceed by induction on n . The case $n = 0$ is trivial. Let the proposition hold for $n < N$. Then for p_N , by Lemma 3.10, we have:

$$p_N^\diamond(a) = - \sum_{j=1}^N a_{Nj} p_{N-j}^\diamond(a) \quad (3.27)$$

and

$$p_N^N(1) \frown a = - \sum_{j=1}^N a_{Nj} p_N^{N-j}(1) \frown a \quad (3.28)$$

By the induction hypothesis the expressions on the right-hand-side coincide. The induction runs. \square

PROPOSITION 3.12. *For every integer $n \geq 0$ the projection morphism $p: \mathbb{P}_X^n \rightarrow X$ belongs to the class \mathfrak{V} .*

Proof. Given $\alpha \in A^*(\mathbb{P}_X^n)$ and $a \in A_*(X)$ one should verify that

$$p_*(\alpha \frown p^!(a)) = p_!(\alpha) \frown a. \quad (3.29)$$

Clearly, both sides of (3.29) are $A^*(X)$ -linear. By the PBT, $A^*(\mathbb{P}_X^n)$ is generated as an $A^*(X)$ -module by the elements ζ^j . Thus, it suffices to check the Proposition just for these elements. From [PS1, Lemma 1.9.1], we have a relation $\zeta^j = i_1^j(1)$ in $A^*(\mathbb{P}_X^n)$, where $i^j: \mathbb{P}_X^{n-j} \hookrightarrow \mathbb{P}_X^n$ is the standard embedding map and the element $\zeta^j \in A^*(\mathbb{P}^n)$ is considered here as lying in $A^*(\mathbb{P}_X^n)$ via the pull-back operator for the projection $\mathbb{P}_X^n \rightarrow \mathbb{P}^n$. Denote by p_j the projection map $\mathbb{P}_X^{n-j} \rightarrow X$. Since $p \circ i^j = p_j$, we have by Proposition 3.7:

$$p_*(\zeta^j \frown p^!(a)) = p_* i_*^j(1 \frown i_!^j p^!(a)) = p_*^j p_j^!(a). \quad (3.30)$$

Using Proposition 3.11, one has:

$$p_*^j p_j^!(a) = p_!^j(1) \frown a = p_! i_1^j(1) \frown a = p_!(\zeta^j) \frown a. \quad (3.31)$$

□

This finishes proof of Theorem 2.6.

4. Proof of The Second Projection Formula

The strategy of the proof of Theorem 2.8 is very similar to one used in the previous section. It is again convenient to introduce a class \mathfrak{W} consisting of projective morphisms $f : Y \rightarrow X$ such that for any $T \in Sm/k$, $\alpha \in A^*(T \times Y)$, and $a \in A_*(X)$ the relation

$$F_!(\alpha)/a = \alpha/f^!(a) \quad (4.1)$$

holds in $A^*(T)$. (Here $F = \text{id} \times f$. Below we use similar notation rules.)

We show that the following classes of morphisms lie in \mathfrak{W} .

- Zero-sections of vector bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$;
- Closed embeddings $i : Y \hookrightarrow X$;
- Projections $p : X \times \mathbb{P}^n \rightarrow X$;

Since the class \mathfrak{W} is closed with respect to composition, this will imply our formula for all projective morphisms.

LEMMA 4.1. *Let \mathcal{V} be a vector bundle over a smooth variety Y and let $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$ be the zero-section of the projection $p : \mathbb{P}(\mathbf{1} \oplus \mathcal{V}) \rightarrow Y$. Then the morphism s belongs to the class \mathfrak{W} .*

Proof. Let $\alpha \in A^*(T \times Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V}))$. Using functoriality of the slant-product, an associativity relation, and formula 2.3, one gets:

$$\begin{aligned} \alpha/s^!(a) &= \alpha/p_*(s_!(1) \frown a) = P^*(\alpha)/(s_!(1) \frown a) \\ &\stackrel{\mathbf{A.1}}{=} (P^*(\alpha) \smile (1 \times s_!(1))) / a = (P^*(\alpha) \smile S_!(1)) / a = S_!(\alpha)/a. \end{aligned} \quad (4.2)$$

(Here the relation $1 \times s_!(1) = S_!(1)$ appears from the base-change property applied to the product with T .) □

PROPOSITION 4.2. *Any closed embedding morphism $i : Y \hookrightarrow X$ of smooth varieties belongs to the class \mathfrak{W} .*

Proof. Denote by $\mathbb{P}(\mathbf{1} \oplus \mathcal{N})$ the projectivization corresponding to the normal bundle $\mathcal{N} = \mathcal{N}_{X/Y}$. It is endowed with the zero-section morphism $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{N})$.

As well as in the proof of Theorem 2.6 our arguments are based on the deformation diagram which obtained from (3.3) by multiplication with a variety $T \in Sm/k$. For convenience, we reproduce this diagram here.

$$\begin{array}{ccccc}
 & & T \times B - T \times Y \times \mathbb{A}^1 & & (4.3) \\
 & & \downarrow K_B & & \\
 T \times \mathbb{P}(\mathbf{1} \oplus \mathcal{N}) & \xrightarrow{K_0} & T \times B & \xleftarrow{K_1} & T \times X \\
 \uparrow S & & \uparrow I_t & & \uparrow I \\
 T \times Y & \xrightarrow{J_0} & T \times Y \times \mathbb{A}^1 & \xleftarrow{J_1} & T \times Y
 \end{array}$$

First of all, we show that $I_t \in \mathfrak{W}$. Namely, we should prove that for any elements $\alpha \in A^*(T \times Y \times \mathbb{A}^1)$ and $a \in A_*(B)$ the relation

$$\alpha/i_t^!(a) = I_t^!(\alpha)/a. \quad (4.4)$$

holds in $A^*(T)$.

Exactly as in the proof of Theorem 2.6 one can rewrite a as a sum $k_*^B(a_B) + k_*^0(a_0)$, where $a_0 \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{N}))$ and $a_B \in A_*(B - Y \times \mathbb{A}^1)$ and obtain the equalities:

$$\alpha/i_t^!(a) = \alpha/i_t^!k_*^0(a_0) = \alpha_0/s^!(a_0), \quad (4.5)$$

where $\alpha_0 = J_0^*(\alpha)$.

Similarly, one gets the relation:

$$I_t^!(\alpha)/a = S_!J_0^*(\alpha)/a_0 = S_!(\alpha_0)/a_0. \quad (4.6)$$

By Lemma 4.1 $\alpha_0/s^!(a_0) = S_!(\alpha_0)/a_0$, which proves (4.4).

Since the map J_1^* is an isomorphism, we can set $\alpha = (J_1^*)^{-1}(\alpha_1) \in A^*(T \times Y \times \mathbb{A}^1)$ and $a = k_*^1(a_1) \in A_*(B)$. Applying Lemma A.3 again, one gets:

$$\alpha_1/i^!(a_1) = J_1^*(\alpha)/i^!(a_1) = \alpha/i_t^!(a) \quad \text{and} \quad (4.7)$$

$$I_!(\alpha_1)/a_1 = I_!J_1^*(\alpha)/a_1 = I_t^!(\alpha)/a. \quad (4.8)$$

Combining these equalities with relation (4.4) proves the proposition. \square

PROPOSITION 4.3. *Let $X, T \in Sm/k$, $p: X \times \mathbb{P}^n \rightarrow X$ be the projection morphism, and $P = \text{id} \times p: T \times X \times \mathbb{P}^n \rightarrow T \times X$. Then for every elements $\alpha \in A^*(T \times X \times \mathbb{P}^n)$ and $a \in A_*(X)$, one has a relation:*

$$\alpha/p^!(a) = P_!(\alpha)/a \quad (4.9)$$

in $A^*(T)$.

Proof. Consider the following commutative diagram with transversal square:

$$\begin{array}{ccccc}
 X \times \mathbb{P}^n & \xleftarrow{i} & X \times \mathbb{P}^{n-r} & \xleftarrow{\bar{q}} & T \times X \times \mathbb{P}^{n-r} \\
 & \searrow^{p=p_0} & \downarrow p_r & & \downarrow P_r \\
 & & X & \xleftarrow{q} & T \times X
 \end{array} \quad (4.10)$$

Clearly, both sides of (4.9) are $A^*(T)$ -linear. So, we may assume that $\alpha = \zeta_{T \times X}^r$. Since $\zeta_{T \times X}^r = I_!(1_{T \times X}) \in A^*(T \times X \times \mathbb{P}^n)$, one has:

$$\begin{aligned}
 \zeta_{T \times X}^r/p^!(a) &= I_!(1_{T \times X})/p^!(a) = 1_{T \times X}/i^!p^!(a) \\
 &= 1/p_r^!(a) = P_r^*(1)/p_r^!(a) = 1/p_*^r p_r^!(a).
 \end{aligned} \quad (4.11)$$

By Proposition 3.11:

$$1/p_*^r p_r^!(a) = 1/(p_r^!(1_X) \frown a) = q^* p_r^!(1)/a. \quad (4.12)$$

Applying the base-change property to the square in the diagram above, we get:

$$q^* p_r^!(1_X) = P_r^!(1_{T \times X}) = P_r!(I_!(1)) = P_r!(\zeta_{T \times X}^r). \quad (4.13)$$

The proposition is proven. \square

A. Some properties of a trace structure

In this Appendix we give a brief description of some useful properties of a trace structure, which are utilized in the paper. Although we make a deal with both cohomological and homological contexts, we present all the results for homology. Cohomological variant is “dual” in the obvious sense and may be found in [PS1]. All the proofs for the homological case not provided below can be found in [Pi].

We, first, give a definition of a transversal square following A.Merkurjev [Me].

DEFINITION A.1. *We call a square*

$$\begin{array}{ccc}
 Y' & \xrightarrow{\bar{f}} & X' \\
 \bar{g} \downarrow & & \downarrow g \\
 Y & \xrightarrow{f} & X
 \end{array}$$

in the category Sm/k transversal if

- a). *it is Cartesian in the category \mathbf{Sch}/k of all schemes over the field k ;*
- b). *the following sequence of tangent bundles over Y' is exact:*

$$0 \rightarrow T_{Y'} \xrightarrow{d\bar{g} \oplus d\bar{f}} \bar{g}^* T_Y \oplus \bar{f}^* T_{X'} \xrightarrow{dg - df} \bar{g}^* f^* T_X \rightarrow 0.$$

It is not hard to check that this definition is accordant to ones given in [PS1, 1.1.2] or [PY, 1.1]. Let us check, for example, that for a closed embedding f condition **b**) implies the isomorphism: $\bar{g}^* \mathcal{N}_{X/Y} \simeq \mathcal{N}_{X'/Y'}$. The short exact sequence above may be viewed as a total complex of the bicomplex:

$$\begin{array}{ccc} 0 & \longrightarrow & \bar{g}^* T_Y \xrightarrow{-df} \bar{g}^* f^* T_X \\ & & \uparrow dg \qquad \qquad \uparrow dg \\ 0 & \longrightarrow & T_{Y'} \xrightarrow{df} f^* T_{X'}. \end{array} \quad (\text{A.1})$$

Since **b**) is exact, the bicomplex is acyclic. On the other hand, it is quasiisomorphic to the two-term complex $\bar{g}^* \mathcal{N}_{X/Y} \leftarrow \mathcal{N}_{X'/Y'}$.

PROPERTY A.2 (Base-change for transversal squares). *For any transversal square as above with projective morphism f the diagram*

$$\begin{array}{ccc} A_*(Y') & \xleftarrow{\bar{f}^!} & A_*(X') \\ \bar{g}_* \downarrow & & g_* \downarrow \\ A_*(Y) & \xleftarrow{f^!} & A_*(X) \end{array}$$

commutes.

COROLLARY A.3. *Suppose, we are given a transversal square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \bar{f} \uparrow & & \uparrow f \\ Y' & \xrightarrow{\bar{g}} & Y \end{array}$$

with projective morphism f . Let $\alpha \in A^(Y)$ and $a \in A_*(X')$. Then the following relations hold:*

i). $f_*(\alpha \frown f^! g_*(a)) = g_* \bar{f}_*(\bar{g}^*(\alpha) \frown \bar{f}^!(a))$

ii). $f_!(\alpha) \frown g_*(a) = g_*(\bar{f}_! \bar{g}^*(\alpha) \frown a)$

Moreover, for a variety $T \in \text{Sm}/k$ and $\beta \in A^(T \times Y)$, we have:*

iii). $\beta / f^! g_*(a) = \bar{G}^*(\beta) / \bar{f}^!(a)$

iv). $F_!(\beta) / g_*(a) = \bar{F}_! \bar{G}^*(\beta) / a$.

Proof. All these relations may be easily obtained using the base-change property. We illustrate it proving the first relation:

$$\begin{aligned} f_*(\alpha \frown f^! g_*(a)) &= f_*(\alpha \frown \bar{g}_* \bar{f}^!(a)) \\ &= f_* \bar{g}_*(\bar{g}^*(\alpha) \frown \bar{f}^!(a)) = g_* \bar{f}_*(\bar{g}^*(\alpha) \frown \bar{f}^!(a)). \end{aligned} \quad (\text{A.2})$$

□

PROPERTY A.4 (Gysin exact sequence). *Let $i: Y \hookrightarrow X$ be a closed embedding and $j: X - Y \hookrightarrow X$ the corresponding open inclusion. Then, the sequence*

$$A_*(X - Y) \xrightarrow{j_*} A_*(X) \xrightarrow{i^!} A_*(Y).$$

is exact.

The following lemma is a “dualization” of “Useful Lemma 1.4.2” from [PS1].

LEMMA A.5 (Homological useful lemma). *Consider the following diagram with transversal square*

$$\begin{array}{ccc} & & X - Y \\ & & \downarrow k^! \\ V & \xrightarrow{k^0} & X \\ \uparrow i & \searrow p & \uparrow q \\ W & \xrightarrow{j} & Y, \end{array}$$

where p is projective, q is a closed embedding, $X - Y$ is the open complement of Y in X , $k_!$ is the corresponding open embedding, $pi = \text{id}$, and the map j induces an isomorphism in homology. Then $\text{Im } k_^0 + \text{Im } k_*^1 = A_*(X)$.*

Proof. Let $x \in A_*(X)$. Since the map j_* is an isomorphism and $i^!p^! = \text{id}$, we can, using the transversal base-change property, lift x up to $\bar{x} = p^!(j_*)^{-1}q^!(x) \in A_*(V)$, such that $q^!k_*^0(\bar{x}) = q^!(x)$. Then, the Gysin exact sequence implies that $k_*^0(\bar{x}) - x \in \text{Im } k_*^1$. \square

PROPERTY A.6 (Projective Bundle Theorem (PBT)). *First, we should introduce the notion of an Euler class. For a line bundle \mathcal{L} over X we set $e(\mathcal{L}) \stackrel{\text{def}}{=} z^*z_!(1)$, where $z: X \rightarrow \mathcal{L}$ is the zero-section (see [PS1, 1.1.4] for details).*

For $X \in \text{Sm}/k$ and a rank n vector bundle $\mathcal{E} \xrightarrow{p} X$ set $\zeta = e(\mathcal{O}_{\mathcal{E}}(1)) \in A^(\mathbb{P}(\mathcal{E}))$. Then the map*

$$\oplus_{i=0}^{n-1} \psi_i: A_*(\mathbb{P}(\mathcal{E})) \xrightarrow{\cong} \bigoplus_{i=0}^{r-1} A_*(X),$$

where $\psi_i = p_ \circ (\zeta^{\smile i} \frown -)$, is an isomorphism.*

B. Chern element and homothety involution

Let $T = \mathbb{P}_k^1$ and take the point 0 as a distinguished point. Let \mathcal{A} be a commutative ring symmetric T -spectrum. For $\lambda \in k^\times$ consider a map $\lambda: T \rightarrow$

T sending $[x: y]$ to $[\lambda x: y]$ (preserving the distinguished point). It defines an involution λ^* on the sphere T -spectrum S [Ya]. For any space X it defines an involution on the cohomology theory $A^{*,*}$ as follows:

$$\epsilon(\lambda)^* = \Sigma_T^{-1} \lambda^* \Sigma_T: A^{*,*}(X) \rightarrow A^{*,*}(X),$$

where $\Sigma_T: A^{*,*}(X) \rightarrow A^{*+2,*+1}(X)$ is the T -suspension isomorphism and Σ_T^{-1} is its inverse. Set

$$\epsilon = \epsilon(-1)^*. \quad (\text{B.1})$$

The following lemma shows that $\epsilon = \text{id}$ for a certain class of T -spectra.

LEMMA B.1. *If \mathcal{A} is equipped with a Chern element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$ then $\epsilon = \text{id}$.*

Proof. We show that for any $\lambda \in k^\times$ one has $\Sigma_T^{-1} \lambda^* \Sigma_T = \text{id}$. Let $i: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be a linear embedding. For $X \in \text{Sm}/k$ consider the projection $p: \mathbb{P}^1 \times X \rightarrow \mathbb{P}^1$. The map

$$(i \times \text{id}_X)^*: A^{*,*}(\mathbb{P}^2 \times X) \rightarrow A^{*,*}(\mathbb{P}^1 \times X)$$

is surjective since 1 and $p^*(\gamma|_{\mathbb{P}^1})$ is a free base of the bigraded $A^{*,*}(X)$ -module $A^{*,*}(\mathbb{P}^1 \times X)$. Now the proof completes as in [HY, Lemma 1.6]

□

REFERENCES

- [Ad] Adams, J.F. *Stable Homotopy and Generalized Homology*. The University of Chicago Press, Chicago and London, 1974, x+373 pp.
- [Fu] Fulton, W. *Intersection theory*. Springer-Verlag, Berlin, 1998, xiv+470 pp.
- [HY] Hornbostel, J. and Yagunov, S. *Rigidity for henselian local rings and \mathbb{A}^1 -representable theories*. 13p. (submitted to Math. Z.)
- [Ja] Jardine, J.F. *Motivic symmetric spectra*. Doc. Math. 5 (2000), 445–553.
- [Me] Merkurjev, A. *Algebraic Oriented Cohomology Theories*. Algebraic Number Theory and Algebraic Geometry, 171–194, Contemp. Math., 300, Amer. Math. Soc., Providence, R.I., 2002.
- [Pi] Pimenov, K. *Traces in Oriented Homology Theories of Algebraic Varieties*. (preprint) www.math.uiuc.edu/K-theory/0724/ 2005.
- [PS1] Panin, I. *Riemann–Roch Theorems for Oriented Cohomology* pp.261–333 in the book J.P.C.Greenlees(ed.) *“Axiomatic, Enriched, and Motivic Homotopy Theory”*. Kluwer Academic Publishers, Netherlands, 2004.
- [PS2] Panin, I. *Push-forwards in Oriented Cohomology Theories of Algebraic Varieties*. (preprint) www.math.uiuc.edu/K-theory/0619/ 2003.

- [PS3] Panin, I. *Oriented cohomology theories on algebraic varieties*. Special issue in honor of H. Bass on his seventieth birthday. Part III. *K-Theory* 30 (2003), no. 3, 265–314.
- [PY] Panin, I. and Yagunov, S. *Rigidity for Orientable Functors*. *Journal of Pure and Applied Algebra*, 172,1(2002) pp.49–77.
- [So] Solynin, A. *Chern and Thom elements in the representable cohomology theory*. Preprint POMI-03/2004, 15p. (available at www.pdmi.ras.ru)
- [Sw] Switzer, R. *Algebraic Topology — Homology and Homotopy*. Springer-Verlag, Berlin, 2002, xii+526 pp.
- [Vo] Voevodsky, V. \mathbb{A}^1 -homotopy theory. Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, pp. 579–604.
- [Vo2] Voevodsky, V. *Cohomological operations in Motivic cohomology*. (unpublished.)
- [Ya] Yagunov S. *Rigidity II: Non-Orientable Case*. Doc. Math., vol.9 (2004) pp.29-40.

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