# Rigidity for equivariant $K$-theory

Serge Yagunov [a], Paul Arne Østvær [b]

[a] *Steklov Mathematical Institute, St. Petersburg, Russia.*
*Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany.*
[b] *Department of Mathematics, University of Oslo, Oslo, Norway.*

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**Abstract**

**Rigidity for equivariant $K$-theory** We extend the classical rigidity results for $K$-theory to the equivariant setting of algebraic group actions. Following Andrei Suslin's pioneering work on ordinary $K$-theory, our results may provide a first step towards an explicit computation of the equivariant $K$-groups of algebraically closed fields.

**Résumé**

**Théorèmes de rigidité classiques pour la $K$-théorie** Nous étendons les théorèmes de rigidité classiques pour la $K$-théorie au cadre équivariant des actions des groupes algébriques. Ces résultats constituent un premier pas vers le calcul explicite des $K$-groupes équivariants des corps algébriquement clos, suivant la stratégie avancée par Andrei Suslin dans le cas des $K$-théorie ordinaires.

## Version française abrégée

Soit $\mathcal{F}$ un foncteur contravariant de la catégorie des schemas lisses projectives de type fini sur un corps infini $k$ à la catégorie des modules. La *propriété de rigidité* est vérifiée pour $\mathcal{F}$ si, pour tout schema $X$ de type spécifié, toutes deux sections $\sigma_0, \sigma_1 \colon \operatorname{Spec} k \to X$ du morphisme structural $X \to \operatorname{Spec} k$ induisent des homomorphismes égaux $\sigma_0^* = \sigma_1^* \colon \mathcal{F}(X) \to \mathcal{F}(\operatorname{Spec} k)$.

Pour un groupe algébrique $\mathsf{G}$ sur $k$ et un $k$-schema $\mathsf{G}$-équivariant de type fini $V$, prenons pour $\mathcal{F}$ le $K$-foncteur $\mathsf{G}$-équivariant aux coefficients finis $\mathsf{K}_*(\mathsf{G}, V \times_k -; \mathbf{Z}/n)$ où $(n, \operatorname{Char} k) = 1$. Nous montrons que la propriété de rigidité est vérifiée pour un tel foncteur $\mathcal{F}$ si et seulement si elle est vérifiée dans le cas particulier où $X$ est une droite affine munie des sections induites par les points $0, 1$.

*Email addresses:* yagunov@gmail.com (Serge Yagunov), paularne@math.uio.no (Paul Arne Østvær).

Une conséquence de ce théorème est que si $V$ est lisse, alors notre foncteur est invariant sous les changements de base suivants : extensions de corps algébriquement clos, et deformations infinitésimales (i.e. morphismes des anneuaux de Hensel locaux à leur corps résiduels).

## 1. Introduction

For $k$ an infinite field and $\mathsf{G}$ an algebraic group over $k$, let $\mathsf{Sm}_k$ denote the category of smooth projective $k$-schemes of finite type and fix some $V$ in the category $\mathsf{Sch}_k^{\mathsf{G}}$ of $\mathsf{G}$-equivariant $k$-schemes of finite type and $\mathsf{G}$-maps. The latter data entails a $\mathsf{G}$-action $\sigma\colon V \times \mathsf{G} \to V$ subject to the usual associative and unital identities for a group action. The abelian subcategory of $\mathsf{G}$-modules on $V$ comprising coherent $\mathcal{O}_V$-modules and its exact subcategory of locally free $\mathcal{O}_V$-modules, a.k.a. $\mathsf{G}$-vector bundles on $V$, gives rise to the $\mathsf{G}$-equivariant $K'$-theory (respectively $K$-theory) of $V$. In the event $\mathsf{G}$ is the trivial group, this reduces to ordinary $K'$- and $K$-theory. Throughout we consider mod-$n$ coefficients for $n$ relatively prime to the exponential characteristic of $k$ and assume that $\mathsf{G}$ acts trivially on the base fields.

**Theorem (Rigidity for rational points).** *The following are equivalent.*

*(1) For every $X$ in $\mathsf{Sm}_k$ with trivial $\mathsf{G}$-action and any two rational points $x_0$ and $x_1$ on $X$,*

$$x_0^* = x_1^*\colon \mathsf{K}_*(\mathsf{G}, V \times_k X; \mathbf{Z}/n) \rightrightarrows \mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n).$$

*(2) The rational points $0$ and $1$ on the affine line $\mathbf{A}^1_k$ with trivial $\mathsf{G}$-action yield equal pullback maps*

$$\mathsf{K}_*(\mathsf{G}, V \times_k \mathbf{A}^1_k; \mathbf{Z}/n) \rightrightarrows \mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n).$$

The four step proof of the theorem consists only of checking that $\mathsf{K}_*(\mathsf{G}, V \times_k -; \mathbf{Z}/n)$ defines a functor with weak transfers on $\mathsf{Sm}_k$ in the sense of [7].

If $V$ is smooth, so that the naturally induced map $\mathsf{K}_*(\mathsf{G}, V \times_k X) \to \mathsf{K}'_*(\mathsf{G}, V \times_k X)$ is an isomorphism for all $X$ in $\mathsf{Sm}_k$, then the second condition in the theorem holds because $K'$-theory is homotopy invariant [12, Theorem 4.1], cf. Lemma 2.3. Although $\mathsf{G}$ acts trivially on $X$ in the previous theorem we do not assume that $\mathsf{G}$ acts trivially on $V$. The following result generalizes [9, Main Theorem].

**Theorem (Rigidity for extensions).** *Suppose $K/k$ is an extension of algebraically closed fields and $(V, \sigma)$ a $\mathsf{G}$-equivariant smooth $k$-scheme of finite type. With the induced $\mathsf{G}$-action $\sigma \times id_K$ on the base change $V_K \equiv V \times_k K$ the natural $\mathsf{G}$-map $V_K \to V$ induces an isomorphism*

$$\mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n) \xrightarrow{\cong} \mathsf{K}_*(\mathsf{G}, V_K; \mathbf{Z}/n).$$

*Proof.* If a noetherian $\mathsf{G}$-scheme $X$ is the inverse limit of a system $\{X_\alpha\}$ of noetherian $\mathsf{G}$-schemes with flat transition maps, then $\mathsf{K}'(\mathsf{G}, X) \cong \operatorname{colim}_\alpha \mathsf{K}'(\mathsf{G}, X_\alpha)$, as Thomason noted in [13, §3.7] (in fact, the category of coherent $\mathsf{G}$-modules on $X$ is the direct limit of the categories of coherent $\mathsf{G}$-modules on the $X_\alpha$'s). Our claim follows now as in the proof of [7, Theorem 1.14] by combining rigidity for rational points and Lemma 2.3 (viewing $\operatorname{Spec} K$ as an inverse limit of smooth affine $k$-schemes with trivial $\mathsf{G}$-actions). □

The last of our main results deals with rigidity for equivariant $K$-theory of Hensel local rings. For non-equivariant results we refer to [1,2] and [11].

**Theorem (Rigidity for Hensel local rings).** *Let $k$ be an infinite field, $(V, \sigma)$ a $\mathsf{G}$-equivariant smooth $k$-scheme, and $X$ a smooth $k$-scheme of finite type. If $P \in X(k)$ is a rational point, denote by $\mathcal{O}^h_{X,P}$ the corresponding Hensel local ring. Then there is a natural isomorphism*

$$\mathsf{K}_*(\mathsf{G}, V \times_k \mathcal{O}^h_{X,P}) \xrightarrow{\cong} \mathsf{K}_*(\mathsf{G}, V).$$

In order to prove this result we basically check that the conditions in [5] are satisfied.

In the study of equivariant $K$-theory over an algebraically closed field $k$ one may ask for an explicit computation of the groups $\mathsf{K}_*(\mathsf{G}, k; \mathbf{Z}/n)$. Based on Suslin's computation of the $K$-groups $\mathsf{K}_*(k; \mathbf{Z}/n)$ [10], we expect this problem is closely related to equivariant topological complex $K$-theory and that an answer involves the representation theory of the algebraic group $\mathsf{G}$.

## 2. Proofs

*Proof.* (*Rigidity for rational points*) For legibility we let $\mathcal{P}^{\mathsf{G}}$ denote $\mathsf{G}$-vector bundles of finite rank, $\mathsf{BQ}$ denote the classifying space of Quillen's $Q$-construction [8], and write

$$\mathcal{F}(-) \equiv \pi_{*+1}\big(\mathsf{BQ}\mathcal{P}^{\mathsf{G}}(V \times_k -); \mathbf{Z}/n\big) \colon (\mathsf{Sm}_k)^{\mathrm{op}} \to \mathbf{Ab}$$

for the functor $\mathsf{K}_*(\mathsf{G}, V \times_k -; \mathbf{Z}/n)$ in the formulation of the theorem. Assuming that (2) holds, we show $\mathcal{F}(-)$ acquires weak transfers for the class $C_{\mathtt{ff}}$ of finite flat maps in the sense of [7].

**Lemma 2.1.** *If $f \colon X \to Y$ is a finite flat map in $\mathsf{Sm}_k$ then the direct image functor induces a transfer map $f_* \colon \mathcal{F}(X) \to \mathcal{F}(Y)$.*

*Proof.* The assumption on $f$ implies that $F = f \times_k \mathrm{id}_V \colon X \times_k V \to Y \times_k V$ is a finite flat map. This shows [3, Théorème 3.2.1, Corollaire 1.3.2] the direct image map $F_*$ is an exact functor on the category of coherent $\mathsf{G}$-sheaves on $X$. In this situation it suffices to verify that $F_*$ preserves locally free sheaves. The latter statement is local, so it suffices to consider affine schemes. Given a ring $R$, a finite flat algebra $S$ over $R$ and a projective $S$-module $P$, we wish to show that $P$ is a projective $R$-module. In effect, we shall verify that the functor $\mathrm{Hom}_R(P, -) \simeq \mathrm{Hom}_S\big(P, \mathrm{Hom}_R(S, -)\big)$ is exact. This follows by combining the standard facts that $S$ is a projective $R$-module according to [6, Theorem 2.9], $P$ is a projective $S$-module and the composition of two exact functors is an exact functor. □

*Remark* 1. *Since we require transfer maps for a very restricted class of maps, we may avoid using higher direct images and* Tor-*formulas as in Quillen's general transfer construction [8, Section 7].*

Secondly, we need to verify the additivity, base change, and normalization conditions formulated in [7].

*Remark* 2. *For our purposes it suffices to check the base change property only for closed embeddings. Therefore, our form of this property is less general than in [7].*

(i) Additivity: For $X = X_0 \coprod X_1$ with corresponding embeddings $i_m \colon X_m \hookrightarrow X$ for $m = 0, 1$ and $f \colon X \to Y$ a map in $\mathsf{Sm}_k$, then
$$f_* = (fi_0)_* i_0^* + (fi_1)_* i_1^*.$$

(ii) Base change: For every cartesian square
$$\begin{array}{ccc} X' & \xhookrightarrow{\tilde{g}} & Y' \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xhookrightarrow{g} & Y \end{array}$$
where $f \in C_{\mathtt{ff}}$ and $g$ is a closed embedding, one has $g^* f_* = \tilde{f}_* \tilde{g}^*$.

(iii) Normalization: If $f$ is the identity map on $k$, then $f_* = \mathrm{id}_{\mathcal{F}(k)}$.

The direct and inverse image functors are $\mathsf{G}$-maps in a natural way, so one can immediately verify that all the isomorphisms appearing in the proof of the base change property below are actually $\mathsf{G}$-isomorphisms. Thus the additivity condition follows immediately by using additivity of the direct image functor.

Next we show that $g^*f_* = \tilde{f}_*\tilde{g}^*$ for the maps in the cartesian diagram. To wit, for a locally free sheaf $\mathcal{S}$ on $Y'$, the adjunction between left adjoint inverse image functors and right adjoint direct image functors furnishes canonical elements $\alpha \in \mathrm{Hom}_{Y'}\big(\mathcal{S}, \tilde{g}_*\tilde{g}^*(\mathcal{S})\big)$ and $\beta \in \mathrm{Hom}_{Y'}\big(f^*f_*(\mathcal{S}), \mathcal{S}\big)$. Moreover, there are canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{Y'}\big(f^*f_*(\mathcal{S}), \tilde{g}_*\tilde{g}^*(\mathcal{S})\big) &\cong \mathrm{Hom}_{X'}\big(\tilde{g}^*f^*f_*(\mathcal{S}), \tilde{g}^*(\mathcal{S})\big)\\ &\cong \mathrm{Hom}_{X'}\big(\tilde{f}^*g^*f_*(\mathcal{S}), \tilde{g}^*(\mathcal{S})\big)\\ &\cong \mathrm{Hom}_{X}\big(g^*f_*(\mathcal{S}), \tilde{f}_*\tilde{g}^*(\mathcal{S})\big).\end{aligned}$$

The image of the composite $\alpha\beta$ in $\mathrm{Hom}_{Y'}\big(f^*f_*(\mathcal{S}), \tilde{g}_*\tilde{g}^*(\mathcal{S})\big)$ under these isomorphisms determines a map of sheaves

$$\widetilde{\alpha\beta}\colon g^*f_*(\mathcal{S}) \longrightarrow \tilde{f}_*\tilde{g}^*(\mathcal{S}).$$

Using the assumptions on $f$ and $g$ one verifies easily that $\widetilde{\alpha\beta}$ is a fiberwise isomorphism. Hence $\widetilde{\alpha\beta}$ is a sheaf isomorphism. This completes the proof of the base-change property.

Finally, the normalization condition holds trivially.

In order to finish the proof it remains to note that the maps appearing in [7, §1] belong to $C_{\mathtt{ff}}$, and moreover that the base change diagrams in loc. cit. are of the type above (with respect to some closed embedding). These conditions hold according to the following elementary result.

**Lemma 2.2.** *If $X$ is a smooth projective curve over a field and $f$ a non-constant rational function on $X$, then the corresponding map $f\colon X \to \mathbf{P}^1$ is a member of $C_{\mathtt{ff}}$.*

*Proof.* This is a special case of [4, III, Proposition 9.7]. □

The proof of rigidity for rational points is now complete. □

The second condition in the rigidity for rational points result serves as a "replacement" for homotopy invariance. A functor $F\colon (\mathsf{Sm}_k^{\mathsf{G}})^{\mathrm{op}} \to \mathbf{Ab}$ is homotopy invariant for $X$ if the canonical projection map $p\colon X \times_k \mathbf{A}^1_k \to X$ induces an isomorphism $p^*\colon F(X) \to F(X \times_k \mathbf{A}^1_k)$. (Here and below $\mathsf{G}$ acts trivially on the affine line $\mathbf{A}^1_k$.)

**Lemma 2.3.** *(1) If $F$ is homotopy invariant for $X$, then the rational points $0$ and $1$ on the affine line over $k$ yield equal pullback maps*

$$F(X \times_k \mathbf{A}^1_k) \rightrightarrows F(X).$$

*(2) If*

$$i_0^* = i_1^*\colon F(Y \times_k \mathbf{A}^1_k) \rightrightarrows F(Y)$$

*holds for $Y = X \times_k \mathbf{A}^1_k$, then $F$ is homotopy invariant for $X$.*

*Proof.* Part (1) holds since the composite map $X \xrightarrow{i} Y \xrightarrow{p} X$ is the identity. In order to prove (2), contemplate the diagram

$$F(X \times_k \mathbf{A}^1_k) \xrightarrow{\mu^*} F(X \times_k \mathbf{A}^1_k \times_k \mathbf{A}^1_k) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} F(X \times_k \mathbf{A}^1_k).$$

Here $\mu^*$ is induced by the product map $\mu\colon \mathbf{A}^1_k \times_k \mathbf{A}^1_k \to \mathbf{A}^1_k$ while the maps $i_0^*$ and $i_1^*$ are induced by the rational points 0 and 1 on $\mathbf{A}^1_k$. By hypothesis, the composite maps in the diagram coincide. The map involving $i_1^*$ is the identity and $i_0\mu$ equals the composite $Y \xrightarrow{p} X \xrightarrow{i} Y$. Hence $i^*$ is inverse to $p^*$. □

As noted in the introduction, the proof of rigidity for extensions is now complete.

*Proof.* (*Rigidity for Hensel local rings*) The Additivity and Normalization properties stated in [5] coincide with the ones in this paper, while the base change diagrams in loc. cit. are also of the type considered here (with respect to some closed embedding), cf. Remark 2. This shows that with our formulation of the base-change property, the approach in [5] can be adopted verbatim. In [5] one also require transfer maps (trace homomorphisms) for finite separable field extensions. It remains to note that such extensions induce maps between schemes in $C_{\mathtt{ff}}$. □

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