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Homology of Bi-Grassmannian Complexes

S. YAGUNOV ⋆
St. Petersburg Branch of the Steklov Mathematics Institute (POMI), Fontanka 27, St. Petersburg, RU-191011, Russia and Department of Mathematics Northwestern University, Evanston, IL 60208-2730, U.S.A.

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Abstract. Homology of bi-Grassmannian complexes with rational coefficients is calculated. Some applications to the homological stabilization of linear groups are given.

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Throughout the last decade, Grassmannian complexes have been playing an important role in the investigation of the homology of general linear groups. (See, for example, [5] and also [6], where complexes similar to Grassmannians play a crucial role.) It can be explained by both their quite geometric structure (these are complexes constructed by considering rational points of an open subset of Grassmannian variety over given field) and the fact that their homology is a coproduct of the relative homology of GL_n's. The natural generalization of Grassmannian complexes are bi-Grassmannian complexes $(G(*,*))$. These are also being used rather widely in research related to the theory of motivic cohomology and polylogarithms (see [2]). However, their homology was still uncalculated.

The following conjecture was formulated by A. Suslin several years ago. Let F be an infinite field and $G(*,*)$ be bi-Grassmannian complex over F. Then

$$H_k(G(*,*)) = \coprod_{0\leqslant p\leqslant (k-1)/2} H_{k-2p-1}(\mathrm{GL}(F)).$$

Subsequently, Suslin found that this conjecture doesn't hold for homology with integral coefficients.

This paper is devoted to the proof of this assertion with rational coefficients.

Also, we prove a similar formula for a truncated complex consisting only of the several bottom rows of bi-Grassmannian complex up to the nth one. This formula is valid in the case where $n!$ is invertible in a coefficient ring.

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1. Definition of Bi-Grassmannian Complexes

Consider an infinite field F. Denote by $\mathrm{Gr}(p,q)$ the set of all planes of dimension p in F^{p+q}. Denote by $\widetilde{G}(p,q)$ the subset of $\mathrm{Gr}(p,q)$ consisting of planes $V \subset F^{p+q}$ defined by the following equivalent conditions:

(a) V intersects any coordinate plane of codimension $\leqslant p$ properly.
(b) The intersection of V and any coordinate plane of codimension p is 0.
(c) If $\pi\colon F^{p+q} \twoheadrightarrow F^p$ is any coordinate projection then the image of V coincides with F^p.

Let us verify the equivalence of these conditions.

(b) $\Leftrightarrow$ (c) There is an one-to-one correspondence between coordinate projections $F^{p+q} \twoheadrightarrow F^p$ and the coordinate planes of codimension p: $\pi \leftrightarrow W = \mathrm{Ker}(\pi)$. On the other hand, we have $\dim \pi(V) = \dim V - \dim(V \cap \mathrm{Ker}(\pi))$, therefore $\pi(V) = F^p \Leftrightarrow V \cap W = 0$.

(a) $\Rightarrow$ (b) Evidently.

(b) $\Rightarrow$ (a) Let U be a coordinate subspace of codimension $k \leqslant p$. Choose on U $p-k$ coordinate functions $x_1, x_2, \ldots, x_{p-k}$ and let

$$W = \{u \in U : x_1(u) = x_2(u) = \cdots = x_{p-k}(u) = 0\}.$$

Then W is a coordinate subspace in F^{p+q} and codim $W = p$. Therefore, $0 = V \cap W = V \cap U|_{x_i=0}$, implies

$$\begin{aligned}\dim(V \cap U) \leqslant p-k &= p + (p+q-k) - (p+q)\\ &= \dim V + \dim U - \dim F^{p+q}.\end{aligned}$$

Thus, the intersection of V and U has the right dimension. $\square$

Introduce two families of maps between the sets $\widetilde{G}(p,q)$. The first ones are projection operators $\mathrm{d}_i\colon \widetilde{G}(p,q) \to \widetilde{G}(p,q-1)$, where d_i is the projection onto the ith coordinate plane of codimension 1. The other family consists of the 'intersection operators' $\partial_j\colon \widetilde{G}(p,q) \to \widetilde{G}(p-1,q)$. More precisely, consider the linear maps

$$F^{p+q-1} \overset{\xi_i}{\hookrightarrow} F^{p+q} \overset{\pi_i}{\twoheadrightarrow} F^{p+q-1},$$
$$\xi_i(x_1, \ldots, x_{p+q-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p+q-1}),$$
$$\pi_i(x_1, \ldots, x_{p+q}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{p+q}).$$

If $V \in \widetilde{G}(p,q)$, then set $\mathrm{d}_i(V) = \pi_i(V), \partial_i(V) = \xi_i^{-1}(V)$.

The previous discussion shows that these operators are well-defined.

LEMMA 1.1.

$$\partial_j\, \mathrm{d}_i = \begin{cases} \mathrm{d}_{i-1}\partial_j, & \text{if } j < i\\ \mathrm{d}_i\partial_{j+1}, & \text{if } j \geqslant i.\end{cases}$$

Proof. Let $W = \partial_j \, \mathrm{d}_i V$, then $W = \xi_j^{-1}(\pi_i(V))$ and for any $w \in W$ there exists such $y \in V$ so that

$$\begin{aligned}&(w_1, \ldots, w_{j-1}, 0, w_j, \ldots, w_{p+q-2})\\&\quad = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{p+q}) \ \ (\xi_j(w) = \pi_i(y)).\end{aligned}$$

The last equality implies that

$$\begin{aligned}&y_j = 0, \qquad \text{if } \ j < i;\\&y_{j+1} = 0, \quad \text{if } \ j \geqslant i.\end{aligned}$$

Omitting y_j in the vector y in the first case or y_{j+1} in the second one, we get a vector z such that $\xi_j(z) = y$ (respectively, $\xi_{j+1}(z) = y$). On the other hand, we have $\pi_{i-1}(z) = w(\pi_i(z) = w)$, therefore $w \in \pi_{i-1}(\xi_j^{-1}(V))$ $(w \in \pi_i(\xi_{j+1}^{-1}(V)))$, i.e. $\partial_j \, \mathrm{d}_i(V) \subset \mathrm{d}_{i-1}\partial_j(V)$ (resp. $\partial_j \, \mathrm{d}_i(V) \subset \mathrm{d}_i\partial_{j+1}(V)$). But dimensions of these spaces are the same, therefore, both inclusions are equalities. □

Further, we will denote by $\mathbb{A}$ a commutative ring with unit. We often denote the groups $H(\ldots, \mathbb{A})$ by $H(\ldots)$. We also will denote by $G(p,q)$ a free $\mathbb{A}$-module $\mathbb{A}[\widetilde{G}(p,q)]$ generated by elements of $\widetilde{G}(p,q)$. Supply the bigraded module $G(*,*)$ by two operations

$$\mathrm{d}\colon G(p,q) \to G(p,q-1), \quad \mathrm{d} = \sum_i (-1)^i \, \mathrm{d}_i$$

and

$$\partial\colon G(p,q) \to G(p-1,q), \quad \partial = \sum_i (-1)^i \partial_i,$$

where d_i and ∂_i are induced by the corresponding maps on $\widetilde{G}(p,q)$.

LEMMA 1.2. *The following equalities hold*: $\mathrm{d}^2 = 0, \partial^2 = 0, \mathrm{d}\partial = -\partial \, \mathrm{d}$.

Proof. We will verify only the last one. The others can be checked in the same way, using the fact that ∂_i and d_i satisfy the simplicial relations.

$$\begin{aligned}\partial d \ &= \ \sum_{j=1}^{p+q-1} \sum_{i=1}^{p+q} (-1)^{i+j} \partial_j \, \mathrm{d}_i\\&= \ \sum_{1 \leqslant i \leqslant j < p+q} (-1)^{i+j} \, \mathrm{d}_i \partial_{j+1} + \sum_{1 \leqslant j < i \leqslant p+q} (-1)^{i+j} \, \mathrm{d}_{i-1}\partial_j\\&= \ \sum_{i=1}^{p+q-1} \sum_{j=i+1}^{p+q} (-1)^{i+j-1} \, \mathrm{d}_i \partial_j + \sum_{i=1}^{p+q-1} \sum_{j=1}^{i} (-1)^{i+j+1} \, \mathrm{d}_i \partial_j = -\mathrm{d}\partial. \quad \square\end{aligned}$$

DEFINITION 1.3. We call the following bicomplex $G(*,*)$ the 'bi-Grassmannian complex'.

$$\begin{tikzcd}
\cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] & \\
G(2,1) \arrow[d, "\partial"'] & G(2,2) \arrow[l, "\mathrm{d}"'] \arrow[d, "\partial"'] & G(2,3) \arrow[l, "\mathrm{d}"'] \arrow[d, "\partial"'] & \cdots \arrow[l, "\mathrm{d}"'] \\
G(1,1) \arrow[d, "\partial"'] & G(1,2) \arrow[l, "\mathrm{d}"'] \arrow[d, "\partial"'] & G(1,3) \arrow[l, "\mathrm{d}"'] \arrow[d, "\partial"'] & \cdots \arrow[l, "\mathrm{d}"'] \\
G(0,1) & G(0,2) \arrow[l, "\mathrm{d}"'] & G(0,3) \arrow[l, "\mathrm{d}"'] & \cdots \arrow[l, "\mathrm{d}"']
\end{tikzcd}$$

Here differentials d and ∂ are defined as above. Further, we fix some integer $n \geqslant 0$ and consider the n-truncated bi-Grassmannian complex $G^n(*,*)$ consisting of rows of $G(*,*)$ from the 0th up to the nth one.

Now we can formulate the main result of the paper.

THEOREM 1.4 (The Main Theorem). *Let F be an infinite field, and $G^n(*,*)$ be the n-truncated bi-Grassmannian complex over F. Assume that $n!$ is invertible in $\mathbb{A}$. Then*

$$H_k(G^n(*,*),\mathbb{A}) = \coprod_{0\leqslant p\leqslant n/2} H_{k-2p-1}(\mathrm{GL}_{n-2p}(F),\mathbb{A}).$$

COROLLARY 1.5. *In the same notation,*

$$H_k(G(*,*),\mathbb{Q}) = \coprod_{0\leqslant p} H_{k-2p-1}(\mathrm{GL}(F),\mathbb{Q}).$$

The bicomplex defined above being quite geometrical is absolutely inconvenient for any calculations. In the next section, we introduce some algebraic objects and a purely algebraically defined bicomplex which is quasi-isomorphic to the bi-Grassmannian complex.

2. Some Auxillary Objects

Let $A = (v_1, v_2, \ldots, v_n)$ be a $m \times n$ matrix over an infinite field F and let $k \leqslant \min(m,n)$ be a nonnegative integer. We will say that A is $W(k)$-matrix if any set of $\min(m,n)$ columns of A containing the columns $v_{n-k+1}, v_{n-k+2}, \ldots, v_n$ is linearly independent. If such a matrix is a $W(0)$-matrix, then we say that it has columns in the general position. Note also that if $m \geqslant n$, then A is a $W(k)$ matrix iff the columns of A are linearly independent (so that, in particular this condition does not really depend on k).

For any $k \leqslant \min(p,q)$ set $WP^{(k)}(p,q)$ to be a free $\mathbb{A}$-module generated by all $p \times q$ $W(k)$-matrices. Consider a complex $WP^{(k)}(p,*)$ having the modules $WP^{(k)}(p,q)$ in dimensions $q > p$ and 0 elsewhere. A differential operator $d^{(k)}$ is given by the formula

$$\mathrm{d}^{(k)}\colon WP^{(k)}(p,q) \to WP^{(k)}(p,q-1),$$

$$\mathrm{d}^{(k)}(A) = \sum_{i=1}^{q-k}(-1)^i(v_1,v_2,\dots,\widehat{v}_i,\dots,v_q).$$

The general linear group $\mathrm{GL}_p(F)$ acts on $WP^{(k)}(p,q)$ by left multiplication. This action commutes with the differential $\mathrm{d}^{(k)}$ and gives each group $WP^{(k)}(p,q)$ a structure of a left GL_p-module. Therefore, we can consider a factor complex ${}_{\mathrm{GL}_p}WP^{(k)}(p,*)$. There is a canonical morphism of complexes

$$\phi_k\colon {}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)[k] \to_{\mathrm{GL}_p} WP^{(0)}(p,*)$$

given as follows. In each orbit U of action of GL_{p+k} on the basis set of the module $WP^{(k)}(p+k,q+k)$ (the set of all $(p+k)\times(q+k)$ $W(k)$-matrices), we can choose an element having the form

$$\begin{pmatrix} M_{p\times q} & 0 \\ * & I_k \end{pmatrix}$$

Moreover, it's clear that the GL_p-orbit of M depends only on U, so that we can define a map ϕ_k using the formula $\phi_k(U) = M \bmod \mathrm{GL}_p$.

THEOREM 2.1. *The map* $\phi_k\colon {}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)[k] \to {}_{\mathrm{GL}_p}WP^{(0)}(p,*)$ *is a quasi-isomorphism.*

LEMMA 2.2. *Denote the complex*

$$0 \leftarrow WP^{(k)}(p+k,k) \xleftarrow{\mathrm{d}^{(k)}} \cdots \xleftarrow{\mathrm{d}^{(k)}} WP^{(k)}(p+k,p+k)$$

by $R^{(k)}(p+k,*)$. *This complex is cyclic up to dimension* $p+k-1$ *and the complex* $WP^{(k)}(p+k,*)[1]$ *is a free* GL_{p+k}*-resolution of the complex* $R^{(k)}(p+k,*)$.

Proof. Obviously, GL_{p+k}-modules $WP^{(k)}(p+k,q)$ are free provided that $q \geqslant p+k$. Thus, we should just check that the sequence

$$0 \leftarrow WP^{(k)}(p+k,k) \xleftarrow{\mathrm{d}^{(k)}} \cdots \xleftarrow{\mathrm{d}^{(k)}} WP^{(k)}(p+k,p+k) \xleftarrow{\mathrm{d}^{(k)}} \\ \xleftarrow{\mathrm{d}^{(k)}} WP^{(k)}(p+k,p+k+1) \xleftarrow{\mathrm{d}^{(k)}} \cdots$$

is exact. Let $w = \Sigma_j a_j(v_{1j},\dots,v_{qj}) \in WP^{(k)}(p+k,q)$ be a cycle. Since the field F is infinite we can choose a general enough vector v_0 such that

$$y = -\sum_j a_j(v_0,v_{1j},\dots,v_{qj}) \in WP^{(k)}(p+k,q+1).$$

Then

$$\begin{aligned} \mathrm{d}y \ &= \ \sum_j a_j(v_{1j},\dots,v_{qj}) + \sum_j a_j \\ &\quad \times \sum_{i=1}^{q-k}(-1)^{i+1}(v_0,v_{1j},\dots,\widehat{v}_{ij},\dots,v_{qj}) = w. \end{aligned}$$ □

Proof of Theorem 2.1. Consider a map $\gamma\colon WP^{(0)}(p,q) \to WP^{(k)}(p+k,q+k)$ given by the formula

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & I_k \end{pmatrix}.$$

We want to show that this map induces an isomorphism

$$\gamma_*\colon H_l({}_{\mathrm{GL}_p}WP^{(0)}(p,*)) \simeq H_{l+k}({}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)).$$

To do so, note that the same map γ also defines a homomorphism of complexes $\gamma\colon R^{(0)}(p,*) \to R^{(k)}(p,*)[k]$. Furthermore, the groups GL_p and GL_{p+k} act on $R^{(0)}$ and $R^{(k)}$, respectively, and the map γ is compatible with the group embedding $\mathrm{GL}_p \hookrightarrow \mathrm{GL}_{p+k}$. Thus, γ induces a homomorphism of spectral sequences $\gamma\colon E \to \overline{E}[0,k]$, where E (resp. $\overline{E}$) is a hyperhomology spectral sequence corresponding to the action of GL_p on $R^{(0)}$ (resp. GL_{p+k} on $R^{(k)}$).

$$\begin{aligned} E^1_{mn} &= H_m(\mathrm{GL}_p, R^{(0)}(p,n)) \Rightarrow H_{m+n}(\mathrm{GL}_p, R^{(0)}(p,*)), \\ \overline{E}^1_{mn} &= H_m(\mathrm{GL}_{p+k}, R^{(k)}(p+k,n)) \Rightarrow H_{m+n}(\mathrm{GL}_{p+k}, R^{(k)}(p+k,*)). \end{aligned}$$

Lemma 2.2 shows that the homomorphism under consideration coincides with the induced map on E^∞-terms

$$\begin{aligned} \gamma_*\colon E^\infty_l \ &= \ H_l(\mathrm{GL}_p, R^{(0)}(p,*)) \simeq H_{l+1}({}_{\mathrm{GL}_p}WP^{(0)}(p,*)) \\ &\to \ H_{l+k+1}({}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)) \\ &\simeq \ H_{l+k}(\mathrm{GL}_{p+k}, R^{(k)}(p+k,*)) = \overline{E}^\infty_{l+k}. \end{aligned}$$

It's sufficient now to show that γ induces an isomorphism on E^1-terms. Namely, let us show that for $n \leqslant p$

$$H_m(\mathrm{GL}_p, R^{(0)}(p,n)) \simeq H_m(\mathrm{GL}_{p-n}) \simeq H_m(\mathrm{GL}_{p+k}, R^{(k)}(p+k, n+k)).$$

The group GL_{p+k} acts transitively on the canonical basis of $WP^{(k)}(p+k, n+k)$. The stabilizer of the element $(e_{p-n+1}, e_{p-n+2}, \ldots, e_{p+k})$ is the affine subgroup

$$\mathrm{Aff}_{p-n,n+k} \stackrel{\text{def}}{=} \begin{pmatrix} \mathrm{GL}_{p-n} & 0 \\ * & I_{n+k} \end{pmatrix}.$$

Using Shapiro's lemma (see, for example, [1]), we get an isomorphism $H_m(\mathrm{GL}_{p+k}, R^{(k)}(p+k, n+k)) \simeq H_m(\mathrm{Aff}_{p-n,n+k})$, but the Theorem 1.11[5] asserts that this group is isomorphic to $H_m(\mathrm{GL}_{p-n})$. (Setting $k = 0$, we have $H_m(\mathrm{GL}_p, R^{(0)}(p,n)) \simeq H_m(\mathrm{GL}_{p-n})$.) It is easy to verify that the resulting isomorphism $H_m(\mathrm{GL}_p, R^{(0)}(p,n)) \simeq H_m(\mathrm{GL}_{p+k}, R^{(k)}(p+k, n+k))$ is induced by the map γ. The isomorphism of E^1-terms of the spectral sequences gives the required isomorphism on the limits. Since $\phi_k\gamma = \mathrm{id}$ and γ is a quasi-isomorphism, we conclude that ϕ_k is a quasi-isomorphism as well. □

Introduce a right action of the symmetric group Σ_k on the complex $WP^{(k)}(p, *)$. Let

$$v = (v_1, \ldots, v_{q-k}, v_{q-k+1}, \ldots, v_q) \in WP^{(k)}(p,q) \quad \text{and} \quad \sigma \in \Sigma_k.$$

We set

$$v\sigma = (v_1, \ldots, v_{q-k}, v_{q-k+\sigma^{-1}(1)}, v_{q-k+\sigma^{-1}(2)}, \ldots, v_{q-k+\sigma^{-1}(k)}).$$

Evidently, this action commutes with the left action of the group GL_p. We will say that vectors $v_{q-k+1}, v_{q-k+2}, \ldots, v_q$ are on the right-hand side of the matrix v and sometimes separate these vectors in formulas by the sign '$|$'. Note that the group Σ_k acts exclusively on the right-hand vectors, whereas the face operators $\partial_i^{(k)}$ act only on vectors on the left-hand side. This allows us to consider factor complexes $WP^{(k)}(p, *)_{\Sigma_k}$ and ${}_{\mathrm{GL}_p}WP^{(k)}(p, *)_{\Sigma_k}$.

LEMMA 2.3. *If $k!$ is invertible in $\mathbb{A}$, then the map ϕ_k induces a canonical quasi-isomorphism*

$$\phi_{*_k}\colon {}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k, *)_{\Sigma_k}[k] \to {}_{\mathrm{GL}_p}WP^{(0)}(p, *).$$

Proof. One can easily see that if $g \in {}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k, q+k)$ and $\sigma \in \Sigma_k$ then $\phi_k(g) = \phi_k(g\sigma)$. Therefore, the following diagram commutes

$$
\begin{tikzcd}
{}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)[k] \arrow[r, "\phi_k"] \arrow[d] & {}_{\mathrm{GL}_p}WP^{(0)}(p,*) \arrow[d, equal] \\
{}_{\mathrm{GL}_{p+k}}WP^{(k)}(p+k,*)_{\Sigma_k}[k] \arrow[r, "\phi_{*_k}"] & {}_{\mathrm{GL}_p}WP^{(0)}(p,*)_{\Sigma_k}
\end{tikzcd}
$$

The second row is the result of application of Σ_k- coinvariant functor to the first one. This functor is exact in the category of $\mathbb{A}$-modules because $H_i(\Sigma_k, \mathbb{A}) = 0$, provided that $i > 0$ (see [1]). Theorem 2.1 tells us that ϕ_k is a quasi-isomorphism. Therefore, ϕ_{*_k} is a quasi-isomorphism too. $\square$

From now on, we assume that n is a fixed nonnegative integer and that $n!$ is invertible in the ring $\mathbb{A}$. Set

$$
S(p,q) \overset{\text{def}}{=} \begin{cases} WP^{(n-p)}(n,q)_{\Sigma_{n-p}}, & 0 \leqslant p \leqslant n, p+q \geqslant n, \\ 0, & \text{otherwise}. \end{cases}
$$

Consider a bigraded module $S(*,*)$ which has $S(p,q)$ in dimension (p,q) if $q > n$ and 0 elsewhere. Supply it by two operators d and ∂, where $\mathrm{d}\colon S(p,q) \to S(p,q-1)$ is induced by the differential operator $\mathrm{d}^{(n-p)}$ above and the operator $\partial\colon S(p,q) \to S(p-1,q)$ is given by the formula $\partial = \Sigma_{i=1}^{p+q-n}(-1)^i\partial_i$, where

$$
\begin{aligned}
&\partial_i(v_1, \ldots, v_{p+q-n} | v_{p+q-n+1}, \ldots, v_q) \\
&\quad = (v_1, \ldots, \widehat{v}_i, \ldots, v_{p+q-n} | v_{p+q-n+1}, \ldots, v_q, v_i).
\end{aligned}
$$

LEMMA 2.4. *$S(*,*)$ is a bicomplex.*

Proof. We have to verify that $\partial^2 = 0, \mathrm{d}\partial = -\partial\,\mathrm{d}$ and $\mathrm{d}^2 = 0$. We will check the first equality. The other ones can be checked in the similar way. Let $x = (v_1, v_2, \ldots, v_q) \in S(p,q)$. We have

$$
\begin{aligned}
\partial^2(x) &= \sum_{i=1}^{q+p-n-1} (-1)^i \partial_i \left(\sum_{j=1}^{q+p-n} (-1)^j (v_1, \ldots, \widehat{v}_j, \ldots \mid \ldots, v_q, v_j) \right) \\
&= \sum_{1 \leqslant i < j \leqslant q+p-n} ((-1)^{i+j} (v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots \mid \ldots, v_q, v_j, v_i) \\
&\quad + (-1)^{i+j-1} (v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots \mid \ldots, v_q, v_i, v_j)) = 0. \qquad \square
\end{aligned}
$$

Let us come back to the bi-Grassmannian complexes. We are going to construct a quasi-isomorphism between ${}_{\mathrm{GL}_n}S(*,*)$ and the bi-Grassmannian complex $G^n(*,*)$. First, consider a map $\psi\colon WP^{(0)}(p,p+q) \to G(p,q)$

$$\psi\begin{pmatrix} a_{11} & \dots & a_{1,p+q} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{p,p+q} \end{pmatrix}$$

$$= [\text{plane with basis } (a_{11},\dots,a_{1,p+q}),\dots,(a_{p1},\dots,a_{p,p+q})].$$

(Recall, that $WP^{(0)}(p,p+q)$ is generated by matrices having columns in general position.) One can easily see that this map ψ defines an isomorphism

$$_{\mathrm{GL}_p}WP^{(0)}(p,p+q) \xrightarrow[\sim]{\psi} G(p,q).$$

Combining it with $\phi_{*_{n-p}}$ we get a map:

$$_{\mathrm{GL}_n}S(p,n+q) \xrightarrow{\phi_{*_{n-p}}} {}_{\mathrm{GL}_p}WP^{(0)}(p,p+q) \xrightarrow[\sim]{\psi} G(p,q).$$

PROPOSITION 2.5. *The composition map $\psi\phi_{*_{n-p}}$ defines a quasi-isomorphism of bicomplexes ${}_{\mathrm{GL}_n}S(*,*)[0,n] \xrightarrow{\sim} G^n(*,*)$.*

The proof is straightforwardly implied by Lemma 2.3 and will be omitted. The previous proposition shows that we can calculate the homology of ${}_{\mathrm{GL}_n}S(*,*)$ instead of the homology of the complex $G^n(*,*)$. The following section is devoted to this calculation.

3. Proof of the Main Theorem

LEMMA 3.1. *If $i > 0, q > n$, then $H_i(\mathrm{GL}_n, S(p,q)) = 0$.*

Proof. The group $S(p,q)$ has a canonical basis consisting of the orbits of action of Σ_{n-p} on $n \times q$ $W(n-p)$-matrices. The group GL_n acts on the canonical basis of $S(p,q)$ by permutations. In each orbit β of its action, we can choose an element which can be presented by the matrix

$$A_\beta = \begin{pmatrix} a_{11} & \dots & a_{1,q-n} & 1 & 0 & \dots & 0 \\ a_{21} & \dots & a_{2,q-n} & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & \dots & a_{n,q-n} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Shapiro's Lemma tells us that

$$H_i(\mathrm{GL}_n, S(p,q)) \simeq \coprod_{\text{all orbits}} H_i(\text{Stab } A_\beta).$$

But for any β, Stab A_β is isomorphic to a subgroup of Σ_{n-p}. Since the order of this group is a divisor of $n!$, we conclude that each group $H_i(\text{Stab } A_\beta)$ is trivial. □

COROLLARY 3.2. *The natural map*

$$H_*(\mathrm{GL}_n, H_{n+1}S(p,*)) \to H_*({}_{\mathrm{GL}_n}S(p,*))[n+1]$$

is an isomorphism.

Proof. Consider two hyperhomology spectral sequences of GL_n with coefficients in the complex $S(p,*)$. The E^2-term of the first spectral sequence is concentrated in the $p+1$th column and has the form $E^2_{l,p+1} = H_l(\mathrm{GL}_n, H_{p+1}(S(p,*)))$. The $\widetilde{E}^1$-term of the second one is concentrated in the 0th row and has the form $\widetilde{E}^1_{0l} = H_0(\mathrm{GL}_n, S(p,l)) =_{\mathrm{GL}_n} S(p,l)$ $(l > n)$. Computation of the $\widetilde{E}^2$-term gives the assertion of the corollary. □

LEMMA 3.3. *For any* $0 \leqslant p \leqslant n$ *the sequence*

$$0 \leftarrow S(p,0) \leftarrow S(p,1) \leftarrow \cdots \leftarrow S(p,n) \leftarrow S(p,n+1) \leftarrow \cdots$$

is exact.

Proof. It follows from Lemma 2.2 and exactness of the Σ_{n-p}-coinvariant functor in our conditions. □

Let us introduce a bicomplex $\overline{S}(*,*)$ given as follows:

$$\begin{tikzcd}
0 & S(n,0) \arrow[l] \arrow[d] & S(n,1) \arrow[l, "d"'] \arrow[d, "\partial"'] & \cdots \arrow[l, "d"'] & S(n,n) \arrow[l, "d"'] \arrow[d, "\partial"'] \\
 & 0 & S(n-1,1) \arrow[l] \arrow[d] & \cdots \arrow[l, "d"'] & S(n-1,n) \arrow[l, "d"'] \arrow[d, "\partial"'] \\
 & & 0 & \cdots \arrow[l] & S(n-2,n) \arrow[l, "d"'] \\
 & & & \vdots & \vdots \\
 & & & 0 & S(0,n) \arrow[l] \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

The differentials d and ∂ are defined here in the same way as in the bicomplex $S(*,*)$. There is a canonical morphism of bicomplexes $S(*,*)[0,1] \to \overline{S}(*,*)$ induced by the differential d. Using Lemma 3.3, it is easy to verify that this map is a quasi-isomorphism. This fact, together with Corollary 3.2, gives us the following proposition:

PROPOSITION 3.4.

$$H_*({}_{\mathrm{GL}_n}S(*,*))[1] \simeq H_*(\mathrm{GL}_n, \overline{S}(*,*)) \qquad \square$$

THEOREM 3.5. *For any $q \leqslant n$ the complex*

$$0 \leftarrow S(n-q,q) \xleftarrow{\partial} \cdots \xleftarrow{\partial} S(n-1,q) \xleftarrow{\partial} S(n,q)$$

is acyclic up to dimension $n-1$.

Proof. Consider some fixed ordered set $V = \{v_1, \ldots, v_q\}$ of linearly independent vectors. Let $C_V(p)$ be the submodule of $S(p,q)$ generated by matrices

$$(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots, v_{\sigma^{-1}(q)}),$$

where $v_j \in V$ and $\sigma \in \Sigma_q$. Since the face operators ∂_j change only the order of columns in a matrix, we can rewrite the complex $S(*,q)$ in the form

$$S(*,q) \simeq \coprod_{\substack{V=\{v_1,v_2,\ldots,v_q\} \\ v_i \text{ linearly independent}}} C_V(*).$$

All summands on the right-hand side are isomorphic to each other. It is sufficient to verify that the complex $C_V(*)$ is acyclic for some set V. Let $T^q(p)$ be the set of all ordered subsets of $\{1,\ldots,q\}$ having cardinality p. Consider the complex

$$D^q_* = (\mathbb{A}[T^q(q)] \xrightarrow{\mathrm{d}} \mathbb{A}[T^q(q-1)] \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathbb{A}[T^q(1)] \xrightarrow{\mathrm{d}} \mathbb{A}),$$

where $\mathrm{d}(\{x_1,\ldots,x_p\}) = \Sigma_{i=1}^p (-1)^i \{x_1,\ldots,\widehat{x}_i,\ldots,x_p\}$. Obviously, $C_V(*)[n-q] \simeq D^q_*$, where $q = \#(V)$. Let us prove that $H_pD^q_* = 0$ for $p < q$. We will make an induction on q. The case $q = 1$ is trivial. Assume that $H_pD^q_* = 0$ for $p < q \leqslant k-1$. Consider the following filtration on the complex D^k_*.

Set $F_lD^k_m$ to be a free $\mathbb{A}$-module denerated by all elements $\{x_1,\ldots,x_m\} \in T^k(m)$ satisfying the condition: if $1 \in \{x_1,\ldots,x_m\}$, then $1 = x_i$ with $1 \leqslant i \leqslant l$, in particular, F_0 is generated by all subsets which don't contain 1.

The term $F_{l/l-1}D^k_m$ of factor filtration is generated by sets

$$\{x_1,\ldots,x_{l-1},1,x_{l+1},\ldots,x_m\}.$$

In the factor complex $F_{l/l-1}D^k$ operators d_i are trivial for $i \leqslant l$, so we have a canonical decomposition

$$F_{l/l-1}D^k_*[l] = \coprod_{T^{k-1}(l-1)} D^{k-l}_*.$$

Using the induction hypothesis, we get $H_pF_{l/l-1}D^k = 0$ for $p < k$. Consider a spectral sequence of the filtered complex converging to the homology of D^k/F_0D^k. We have $E^1_{lm} = H_{l+m}(F_{l/l-1}D^k)$. But as just was proved, it is 0 for $l + m < k$. Consider now a long exact sequence:

$$\cdots \to H_{p+1}(D^k/F_0D^k) \to H_p(F_0D^k) \xrightarrow{\alpha} H_p(D^k) \to H_p(D^k/F_0D^k) \to \cdots$$

One easily checks that the inclusion $F_0D^k \hookrightarrow D^k$ is homotopic to 0 and, hence, the above map α is trivial. Thus, $H_p(D^k) \hookrightarrow H_p(D^k/F_0D^k)$ for $p < k$ and since $H_p(D^k/F_0D^k) = 0$, we conclude that $H_p(D^k) = 0$. This completes the proof of the theorem. $\square$

LEMMA 3.6. *Let* $x \in S(n,q), 0 \leqslant q \leqslant n$. *If* $\partial x = 0$ *then* $\mathrm{d}x = 0$.

Proof. There is a homomorphism $\alpha\colon S(n-1,q) \to S(n,q-1)$

$$\alpha((v_1, v_2, \ldots, v_{q-1} \mid v_q)) = (v_1, v_2, \ldots, v_{q-1})$$

satisfying the formula: $\alpha\partial = \mathrm{d}$. $\square$

PROPOSITION 3.7.

$$H_*(\mathrm{GL}_n, \overline{S}) = \coprod_{0\leqslant j\leqslant n} H_*(\mathrm{GL}_n, \mathrm{Ker}(S(n,j) \to S(n-1,j)))[-n-j]$$

Proof. Denote $\mathrm{Ker}(S(n,j) \xrightarrow{\partial} S(n-1,j))$ by $K(j)$. There is a natural map between the complex

$$0 \to K(0) \xleftarrow{\mathrm{d}} K(1) \xleftarrow{\mathrm{d}} \cdots \xleftarrow{\mathrm{d}} K(n)$$

and $\overline{S}[n,0]$ induced by ∂. Theorem 3.5, shows this map is a quasi-isomorphism. Using Lemma 3.6, we see that the differential of the complex $K(*)$ is trivial. Because of that, $H_*(\mathrm{GL}_n, \overline{S}) = \amalg_{0\leqslant j\leqslant n} H_*(\mathrm{GL}_n, K(j))[-n-j]$. $\square$

LEMMA 3.8. *For any* $0 \leqslant p, q \leqslant n$ *and* $p + q \geqslant n$

$$H_*(\mathrm{GL}_n, S(p,q)) \simeq H_*(\mathrm{GL}_{n-q})$$

Proof. Since $q \leqslant n$, the group GL_n acts transitively on the canonical basis of $S(p,q)$. Shapiro's lemma tells that the homology under consideration equals the homology of the stabilizer of an arbitrary basis element. But the stabilizer of the typical element $(e_{n-q+1}, e_{n-q+2}, \ldots, e_n)$ has the form

$$\mathrm{Stab} = \begin{pmatrix} \mathrm{GL}_{n-q} & 0 & 0 \\ * & I_{p+q-n} & 0 \\ * & * & \widehat{\Sigma}_{n-p} \end{pmatrix},$$

where $\widehat{\Sigma}_{n-p} \subset \mathrm{GL}_{n-p}$ is a subgroup of permutation matrices. Using Theorem 1.11 [5] and the fact that the group $\widehat{\Sigma}_{n-p}$ doesn't have homology except H_0, we obtain $H_*(\mathrm{Stab}) \simeq H_*(\mathrm{GL}_{n-q})$. $\square$

LEMMA 3.9. *Let* $0 \leqslant p, q \leqslant n$ *and* $p+q \geqslant n$. *Then the map*

$$\partial_*\colon H_*(\mathrm{GL}_{n-q}) \simeq H_*(\mathrm{GL}_n, S(p,q)) \to H_*(\mathrm{GL}_n, S(p-1,q)) \simeq H_*(\mathrm{GL}_{n-q})$$

is trivial if $p+q-n$ *is even and coincides with the identity map otherwise.*

Proof. The map ∂_* can be rewritten as $\partial_* = \Sigma(-1)^i \partial_{i*}$, where ∂_{i*} is the map between homology groups induced by the ith face operator. It is sufficient to prove that ∂_{i*} is the identity map. Consider the category of pairs (group, module). The isomorphism given in the previous lemma is induced by the morphism (i,u), where i is the canonical embedding and $u(1) = (e_{n-q+1}, \ldots, e_n)_{\Sigma_{n-p}}$. The map ∂_i acts on the frame $(e_{n-q+1}, \ldots, e_n)$ as some permutation matrix α. This matrix commutes with $i(\mathrm{GL}_{n-q})$. Therefore, we have $(\alpha i \alpha^{-1}, \alpha u) = (i, \alpha u)$. But inner automorphisms act trivially in homology so we get an equality $(i,u)_* = (i,\alpha u)_*$ which completes the proof. $\square$

PROPOSITION 3.10. *For any* $0 \leqslant j \leqslant n$, *we have*

$$H_k(\mathrm{GL}_n, K(j)) \simeq \begin{cases} 0, & j \text{ is odd} \\ H_k(\mathrm{GL}_{n-j}), & \text{otherwise} \end{cases}$$

Proof. Let us consider two hyperhomology spectral sequences of GL_n with coefficients in $S(*,j)$. The first one has the form

$$E^2_{lk} = E^\infty_{lk} = \begin{cases} H_k(\mathrm{GL}_n, K(j)), & l = n \\ 0, & \text{otherwise}. \end{cases}$$

The second one has the first term $\widetilde{E}^1_{\mathrm{pr}} = H_p(\mathrm{GL}_n, S(r,j)) \simeq H_p(\mathrm{GL}_{n-j})$ and the differential d^1 has the form (Lemma 3.9)

$$d^1 = \begin{cases} 0, & \text{if } r+j-n \text{ is even} \\ \mathrm{id}, & \text{otherwise}. \end{cases}$$

Therefore,

$$\widetilde{E}^{\infty}_{\text{pr}} = \widetilde{E}^{2}_{\text{pr}} = \begin{cases} H_p(\mathrm{GL}_n, S(n,j)) \simeq H_p(\mathrm{GL}_{n-j}), & j \text{ even; } r = n \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Proof of Theorem 1.4. Consider the following chain of isomorphisms

$$\begin{aligned} \coprod_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} H_{k-i}(\mathrm{GL}_{n-i}) &\overset{\text{(prop. 3.10)}}{\simeq} \coprod_{0\leqslant j\leqslant n} H_{k-j}(\mathrm{GL}_n, K(j)) \\ &\overset{\text{(prop. 3.7)}}{\simeq} H_{n+k}(\mathrm{GL}_n, \overline{S}(*,*)) \overset{\text{(prop. 3.4)}}{\simeq} H_{n+k+1}(_{\mathrm{GL}_n} S(*,*)) \\ &\overset{\text{(prop. 2.5)}}{\simeq} H_{k+1}(G^n(*,*)) \end{aligned} \qquad \square$$

Now we consider the behavior of truncated bi-Grassmannian complexes for different values of n. (Up to the end of the paper, any object related to the complex G^n will be supplied by superscript n.)

THEOREM 3.11. *For any $n \geqslant 0$, the following diagram commutes.*

$$\begin{array}{ccc} \coprod_{0\leqslant i\leqslant n/2} H_{k-2i}(\mathrm{GL}_{n-2i}) & \xrightarrow{\sim} & H_{k+1}(G^n(*,*)) \\ \downarrow & & \downarrow \\ \coprod_{0\leqslant i\leqslant (n+1)/2} H_{k-2i}(\mathrm{GL}_{n+1-2i}) & \xrightarrow{\sim} & H_{k+1}(G^{n+1}(*,*)). \end{array}$$

Here the right vertical arrow is induced by the embedding of bi-Grassmannian complexes and the left one by the natural embeddings $\mathrm{GL}_{n-2i} \hookrightarrow \mathrm{GL}_{n+1-2i}$.

Proof. Shapiro's lemma implies that the map induced by the inclusion

$$\lambda\colon K^n(n-k) \to K^{n+1}(n-k) \qquad \left(\lambda(A) = \begin{pmatrix} A \\ 0 \end{pmatrix}\right)$$

makes the diagram

$$\begin{array}{ccc} H_i(\mathrm{GL}_k) & \longrightarrow & H_i(\mathrm{GL}_n, K^n(n-k)) \\ \downarrow & & \downarrow \lambda \\ H_i(\mathrm{GL}_{k+1}) & \longrightarrow & H_i(\mathrm{GL}_{n+1}, K^{n+1}(n-k)) \end{array}$$

commutative. The map λ can be extended to the map of bicomplexes $\overline{S}^n \xrightarrow{\lambda} \overline{S}^{n+1}[1,0]$. Consider another map of these bicomplexes $\overline{S}^n \xrightarrow{\mu} \overline{S}^{n+1}[0,1]$ given as follows. For $A \in S^n(p,q)$ set

$$\mu(A) = \begin{pmatrix} & A & & 0 \\ & & & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \in S^{n+1}(p,q+1).$$

One can easily see that the map $s\colon \overline{S}^n \to \overline{S}^{n+1}[1,1]$

$$A \mapsto \left(\begin{array}{cccc|ccc} a_{11} & \dots & a_{1,p+q-n} & 0 & a_{1,p+q-n+1} & \dots & a_{1,q} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n,p+q-n} & 0 & a_{n,p+q-n+1} & \dots & a_{n,q} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{array}\right)$$
$$\in S^{n+1}(p+1,q+1)$$

gives a chain homotopy between λ and μ.

The map μ can be extended to the map of resolutions $S^n \xrightarrow{\mu} S^{n+1}[0,1]$ and, finally, we have a commutative diagram of bicomplexes:

$$\begin{tikzcd} {}_{\mathrm{GL}_n}S^n[0,n] \arrow[r] \arrow[d, "\mu"'] & G^n \arrow[d] \\ {}_{\mathrm{GL}_{n+1}}S^{n+1}[0,n+1] \arrow[r] & G^{n+1} \end{tikzcd}$$

whose rows are quasi-isomorphisms. (See Proposition 2.5.) □

Theorem 3.11 shows that we may really pass to direct limits on n and deduce Corollary 1.5 from Theorem 1.4.

As an immediate application of Theorem 3.11, we can also obtain the following case of the stabilisation theorem for the linear groups (see [3]). Let us assume that $(n+1)!$ is invertible in the coefficient ring $\mathbb{A}$. Consider the natural embedding $G^n \hookrightarrow G^{n+1}$. This map gives us an isomorphism in homology groups up to degree $n-1$

$$H_k(G^n, \mathbb{A}) \simeq H_k(G^{n+1}, \mathbb{A}) \quad (k < n)$$

and an epimorphism

$$H_n(G^n, \mathbb{A}) \twoheadrightarrow H_n(G^{n+1}, \mathbb{A})$$

COROLLARY 3.12. *The map* $H_k(\mathrm{GL}_n, \mathbb{A}) \to H_k(\mathrm{GL}_{n+1}, \mathbb{A})$ *induced by the natural embedding* $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ *is an isomorphism if* $k < n$ *and epimorphism if* $k = n$*, provided that* $(n+1)!$ *is invertible in the coefficient ring.*

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