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Rigidity for orientable functors

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Abstract

In the following paper we introduce the notion of orientable functor (orientable cohomology theory) on the category of projective smooth schemes and define a family of transfer maps. Applying this technique, we prove that with finite coefficients orientable cohomology of a projective variety is invariant with respect to the base-change given by an extension of algebraically closed fields. This statement generalizes the classical result of Suslin, concerning algebraic K-theory of algebraically closed fields. Besides K-theory, we treat such examples of orientable functors as etale cohomology, motivic cohomology, algebraic cobordism. We also demonstrate a method to endow algebraic cobordism with multiplicative structure and Chern classes. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Eighteen years ago Suslin published the following remarkable result concerning the algebraic K-theory of algebraically closed fields [18].

Theorem 0.1 (Suslin [18]). *Let $F_0 \subset F$ be an extension of algebraically closed fields. Then, for $(p, Char\,F) = 1$ the natural map*

$$K_*(F_0, \mathbb{Z}/p) \to K_*(F, \mathbb{Z}/p)$$

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is an isomorphism. (*Here and below Char denotes "exponential" characteristic i.e.* $Char\,k = char\,k = p$ *for* $p > 0$ *and* $Char = 1$ *for a field of characteristic* 0.)

Later, in papers of Gillet–Thomason [5] and Gabber [4] similar results were established for the case of strict Henselization at a smooth point of an algebraic variety and for Hensel pairs.

All the proofs are based on the concept of "rigidity" for K-theory, which appeared in [18].

Further developments in this field were made in a paper of Suslin–Voevodsky [19] in 1996. There the K-functor was replaced by a "homotopy invariant functor with transfers". It was shown that for such a class of functors similar rigidity property also holds. Unfortunately, it is usually a very non-trivial question whether a functor is a "functor with transfers".

The aim of the present paper is to prove the rigidity property (and, therefore, an analogue of the Suslin's theorem as well) for "orientable" functors. Roughly speaking, we call a multiplicative cohomology theory on schemes orientable if it has Chern classes which satisfy the standard list of properties (including the "projective bundle" theorem). We actually prove that an orientable functor can be endowed with transfers. (Our definition of "functors with transfers" is different from one in [19].) In order to construct the desired transfers, we "replant" the topological constructions from [2,17] into the "soil" of algebraic geometry.

The orientability seems to be a very natural and easy-to-check property, which holds for a rather wide class of theories. In Section 6 we give a brief exposition of such theories. Besides the K-functor, we shall consider étale cohomology, motivic cohomology, and algebraic cobordism and show that they are orientable in our sense. The latter example seems to be the most interesting one.

We learned the concept of orientable spectrum in the algebraic context from lectures given by Fabien Morel during the workshop in Münster in June 1999. We are in debt to him and to Andrei Suslin for inspiring discussions during our work.

Most of the results presented in this paper were obtained during the stay of the second author at Max-Planck-Institut für Mathematik in Bonn. The current paper is the refined version of preprint #69, 2000 of MPI. The second author is very grateful to the Institut for its hospitality and the very nice opportunity to work there for eleven months. Both authors are grateful to the TMR Network grant ERB FMRX CT-97-0107 for the travel support during their work.

1. Functors with weak transfers

We are starting with some basic definitions. Let Sch/k be a category of schemes of finite type over a field k. We denote by Sm/k a full subcategory of Sch/k whose objects are smooth projective algebraic varieties over k.

Definition 1.1. Consider the Cartesian square

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

of objects and morphisms in the category Sm/k with regular closed embeddings $X \hookrightarrow Y$ and $X' \hookrightarrow Y'$. We call this square *transversal* if the natural map of the corresponding normal bundles

$$g^*(\mathcal{N}_{Y/X}) \to \mathcal{N}_{Y'/X'}$$

is an isomorphism.

Example 1.2. Flatness of the morphism f in the square above implies the transversality of the square. (See [3, Appendix B.7.4].)

Example 1.3. Let

$$\begin{tikzcd} \coprod_{y_i/x} y_i \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ x \arrow[r] & X \end{tikzcd}$$

be a Cartesian square, where $f: Y \to X$ is a morphism of smooth curves, étale over the point x. Then this square is transversal.

Let $\mathscr{C} \subset Mor(Sm/k)$ denote either the class of all projective morphisms or the class of finite projective morphisms. We use a term $\mathscr{C}$-morphism to specify a morphism of this class.

Let $\mathscr{F}: (Sm/k)^{\circ} \to \mathscr{A}b$ be a contravariant functor from the category Sm/k to an abelian category $\mathscr{A}b$. Let us also assume that for every $\mathscr{C}$-morphism $f: X \to Y$ endowed with the decomposition $X \overset{\tau}{\hookrightarrow} \mathbb{P}^n \times Y \xrightarrow{p_Y} Y$ there exists a corresponding map $f_*^{\tau}: \mathscr{F}(X) \to \mathscr{F}(Y)$ called *the transfer map* and the family of transfer maps satisfies the three properties below.

Remark 1.4. From now on we accept the following convention. Let $X \overset{\tau}{\hookrightarrow} Y \times \mathbb{P}^n \to Y$ be a $\mathscr{C}$-morphism with fixed decomposition. Let f' be a base-change of f with respect to a morphism g. We shall consider only decompositions of the map f' which make all squares in the diagram below Cartesian.

$$\begin{tikzcd} X' \arrow[r, hook, "\tau'"] \arrow[d] \arrow[rr, bend left, "f'"] & Y' \times \mathbb{P}^n \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X' \arrow[r, hook, "\tau"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{P}^n \arrow[r] & Y \end{tikzcd}$$

This actually means we should choose the embedding $\tau' : X' \hookrightarrow Y' \times \mathbb{P}^n$ canonically, setting τ' be a base-change of τ with respect to $g \times id$.

Following this convention, we will usually omit embedding maps in the notation for transfers.

Let us now write down the list of properties we want our functor $\mathscr{F}$ to satisfy.

Property 1.5 (Base change). *For any $\mathscr{C}$-morphism $f : X \to Y$ endowed with a decomposition $X \overset{\tau}{\hookrightarrow} Y \times \mathbb{P}^n \to Y$ and a morphism $g : Y' \to Y$ such that the square*

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, hook, "\tau'"] \arrow[d, "g'"'] & Y' \times \mathbb{P}^n \arrow[d, "g \times id"] \\ X \arrow[r, hook, "\tau"] & Y \times \mathbb{P}^n \end{tikzcd}$$

is transversal, the diagram:

$$\begin{tikzcd} \mathscr{F}(X') \arrow[r, "f'_*"] & \mathscr{F}(Y') \\ \mathscr{F}(X) \arrow[r, "f_*"'] \arrow[u, "g'^*"] & \mathscr{F}(Y) \arrow[u, "g^*"'] \end{tikzcd}$$

commutes.

Property 1.6 (Finite additivity). *Let $X = X_0 \sqcup X_1$, $j_m : X_m \hookrightarrow X$ $(m = 0, 1)$ be embedding maps, and $f : X \to Y$ be a $\mathscr{C}$-morphism. Setting $f_m = f j_m$, we have*:

$$f_{0,*} j_0^* + f_{1,*} j_1^* = f_*.$$

Property 1.7 (Normalization). *Let $X = Y = Spec(k)$ and f be the identity map. Then for any decomposition $X \overset{\tau}{\hookrightarrow} \mathbb{P}^n \to Y$ the map $f_* : \mathscr{F}(X) \to \mathscr{F}(Y)$ is the identity.*

Definition 1.8. A functor $\mathscr{F}$: $(Sm/k)^\circ \to \mathscr{A}b$ is called a *functor with weak transfers for the class $\mathscr{C}$* if it can be endowed with the family of transfer maps $\{f_*\}_{f \in \mathscr{C}}$, satisfying Properties 1.5–1.7.

Definition 1.9. A functor $\mathscr{F}\colon (Sm/k)^{\circ} \to \mathscr{A}b$ is called a *homotopy invariant functor* if for every $X \in Sm/k$ the map $p_X^* : \mathscr{F}(X) \to \mathscr{F}(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \xrightarrow{p_X} X$ is an isomorphism.

We prove here one property of homotopy invariant functors, which will be used in subsequent sections.

Proposition 1.10. *Let $X \in Sm/k$ and $\mathscr{E} \xrightarrow{p} X$ be a vector bundle of dimension n over X. Then the map $\mathscr{F}(X) \xrightarrow{p^*} \mathscr{F}(\mathscr{E})$ is an isomorphism.*

Proof. Let $s : X \to \mathscr{E}$ be the zero-section of the considered bundle. We have $s^* p^* = (ps)^* = id$, which shows p^* is a monomorphism.

Consider now two embeddings: $i_0, i_1 : \mathscr{E} \to \mathscr{E} \times \mathbb{A}^1$ given by points $\{0\}$ and $\{1\}$ of $\mathbb{A}^1$. Let $pr : \mathscr{E} \times \mathbb{A}^1 \to \mathscr{E}$ be the projection map. We get: $pr \circ i_m = id_{\mathscr{E}}$, therefore, $i_0^* \circ pr^* = i_1^* \circ pr^*$. Since pr^* is an isomorphism, $i_0^* = i_1^*$.

Let now $\{U_i = Spec\, R_i\}$ be an affine open covering of the scheme X such that the bundle $\mathscr{E}$ is trivial over each chart U_i. (So, $\mathscr{E}|_{U_i} = Spec\, R_i[x_1, \ldots, x_n]$.) Let us define maps $\bar{\varphi}_i : R_i[x_1, \ldots, x_n] \to R_i[x_1, \ldots, x_n, t]$, setting $\bar{\varphi}_i(x_k) = tx_k$ $(k = 1, \ldots, n)$. These maps can be patched together, forming the morphism of schemes $\varphi : \mathscr{E} \times \mathbb{A}^1 \to \mathscr{E}$.

One can easily check that $\varphi i_1 = id_{\mathscr{E}}$ and $\varphi i_0 = sp$. Therefore, we have: $p^* s^* = i_0^* \varphi^* = i_1^* \varphi^* = (\varphi i_1)^* = id$. Summarizing the observations above, we see that p^* is an isomorphism inverse to s^*. $\square$

From now on we consider the case of algebraically closed base field k. Let $\mathscr{F}$: $(Sm/k)^{\circ} \to \mathscr{A}b$ be a contravariant functor and X be a smooth affine curve over k. We can construct a map $\Phi : Div(X) \to Hom(\mathscr{F}(X), \mathscr{F}(k))$ defined on canonical generators by the formula: $[x] \mapsto x^*$, where $x^* : \mathscr{F}(X) \to \mathscr{F}(k)$ is the pull-back map, corresponding to the point $x \in X(k)$.

Below we will need a notion of the relative Picard group $Pic(\bar{X}, X_\infty)$, where $\bar{X}$ is the smooth compactification of X and X_∞ denotes the complement of X in $\bar{X}$. Let us recall that this group is generated by divisors D of the curve X and two divisors D and D' are considered to be equivalent if and only if there exists a rational function f on X regular in a neighborhood of X_∞ such that $f|_{X_\infty} = 1$, $div_0(f) = D$, and $div_\infty(f) = D'$.

Theorem 1.11. *Let X be a smooth curve over an algebraically closed field k and let $\mathscr{F}$ be a homotopy invariant functor with weak transfers for the class of finite projective morphisms. Then, in the above notation, the map Φ can be decomposed in the following way*:

$$\begin{tikzcd}
Div(X) \arrow[rr, "\Phi"] \arrow[dr, "\Omega"'] & & Hom(\mathscr{F}(X), \mathscr{F}(k)) \\
& Pic(\bar{X}, X_\infty), \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

where the map Ω is the canonical homomorphism.

Proof. Since the field k is algebraically closed, the group $Pic(\bar{X}, X_\infty)$ is generated by unramified divisors on the curve X (i.e. the divisors which have the form $\sum_i \pm[x_i]$, where x_i are pair wise different points of X). All the relations between unramified divisors come from ones of the following type: $D \sim D'$ if and only if there exists a rational function on X regular in a neighborhood of X_∞ and such that $f|_{X_\infty} = 1$, $div_0(f) = D$, and $div_\infty(f) = D'$. Let us define $\Psi(D) = \Phi[\Omega]^{-1}(D)$. In order to check this map is well-defined, we need to prove that $\Phi(D) = \Phi(D')$ for equivalent divisors $D \sim D'$. One may assume that $Supp(D) \cap Supp(D') = \emptyset$. Let us take a function $f \in k(X)$ such that $f|_{X_\infty} = 1$, $div_0(f) = D$, and $div_\infty(f) = D'$.

Denote by X^0 the open locus of $\bar{X}$ where $f \neq 1$. By the choice of the function f, we have: $X^0 \subset X$ and the morphism $f : X^0 \to \mathbb{P}^1 - \{1\} = \mathbb{A}^1$ is finite. Moreover, since the corresponding divisors are unramified, the map f is étale over the points $\{0\}$ and $\{\infty\}$. Consider the diagram

$$\begin{tikzcd}
\bigoplus_{x/\infty} \mathscr{F}(x) \arrow[dr, "\sum_{x/\infty} f_{x,*}"'] & \mathscr{F}(D') \arrow[l] \arrow[d, "f_{\infty,*}"] & \mathscr{F}(X^0) \arrow[l] \arrow[r] \arrow[d, "f_*"] & \mathscr{F}(D) \arrow[r] \arrow[d, "f_{0,*}"] & \bigoplus_{x/0} \mathscr{F}(x) \arrow[dl, "\sum_{x/0} f_{x,*}"] \\
 & \mathscr{F}(k) & \mathscr{F}(\mathbb{A}^1) \arrow[l, "i_\infty^*"'] \arrow[r, "i_0^*"] & \mathscr{F}(k) &
\end{tikzcd} \tag{1.1}$$

In this diagram $f_{0,*}$ ($f_{\infty,*}$) is the transfer map corresponding to the morphism f_0 (f_∞), which is a restriction of the morphism f to the divisor D (D', respectively). The maps i_0^*, i_∞^* are induced by the points $\{0\}$ and $\{\infty\}$ of $\mathbb{P}^1$.

Finally, all summations in the very right (left) part of the diagram are over the finite set of all points in the preimage of $\{0\}$ ($\{\infty\}$).

Example 1.3 and Property 1.5 of functors with weak transfers show that two central squares of the diagram above commute. Property 1.6 implies that the triangles commute as well. Finally, from Property 1.7 we can see that for each point x over $\{\infty\}$ or over $\{0\}$ one has relation: $f_{x,*} = id$.

Both the maps i_∞^* and i_0^* are isomorphisms which are inverse to the isomorphism $p^* : \mathscr{F}(k) \to \mathscr{F}(\mathbb{A}^1)$ (by the homotopy invariance property). So that, we have

$$\sum_{x/\infty} x^* = \sum_{x/0} x^* : \mathscr{F}(X^0) \to \mathscr{F}(k). \tag{1.2}$$

This clearly implies

$$\Phi(D') = \sum_{x/\infty} x^* = \sum_{x/0} x^* = \Phi(D) : \mathscr{F}(X) \to \mathscr{F}(k). \qquad \square \tag{1.3}$$

Corollary 1.12. *Let us assume, in addition to the hypothesis of Theorem* 1.11, *that there exists an integer n relatively prime to $Char(k)$ such that $n\mathscr{F}(Y) = 0$ for any $Y \in Sm/k$. Then the map Ψ can be passed through the degree map $Pic(\bar{X}, X_\infty) \xrightarrow{deg} \mathbb{Z}$. Namely, if D, D' are two divisors of the same degree, then $\Psi(D) = \Psi(D') : \mathscr{F}(X) \to \mathscr{F}(k)$.*

Proof. We have an exact sequence:

$$0 \to Pic(\bar{X}, X_\infty)^\circ \to Pic(\bar{X}, X_\infty) \xrightarrow{deg} \mathbb{Z} \to 0. \tag{1.4}$$

Since the group $Pic(\bar{X}, X_\infty)^\circ$ is n-divisible over an algebraically closed field of characteristic relatively prime to n, we have: $\Psi|_{Pic(\bar{X}, X_\infty)^\circ} = 0$. The corollary follows. □

Theorem 1.13 (The Rigidity Theorem). *Let $\mathscr{F} : (Sm/k)^\circ \to \mathscr{A}b$ be a contravariant homotopy invariant functor with weak transfers for the class of finite projective morphisms. Assume that the field k is algebraically closed and $n\mathscr{F} = 0$ for some integer n relatively prime to $Char\, k$. Then for every smooth affine variety T and for any two k-rational points $t_1, t_2 \in T(k)$ the induced maps $t_1^*, t_2^* : \mathscr{F}(T) \to \mathscr{F}(k)$ coincide.*

Proof. There exists an irreducible smooth curve $X \subset T$ joining the points t_1 and t_2 (see [15, p. 56]). The statement now follows from Corollary 1.12. □

Extend the functor $\mathscr{F}$ to pro-objects in the category Sm/k by setting

$$\mathscr{F}\left(\lim_{\leftarrow}(X_i)\right) = \lim_{\rightarrow} \mathscr{F}(X_i). \tag{1.5}$$

Theorem 1.14. *Let $\mathscr{F} : (Sm/k)^\circ \to \mathscr{A}b$ be a homotopy invariant functor with weak transfers for the class of finite projective morphisms. Let n be an integer relatively prime to $Char(k)$ such that for every $Y \in Sm/k$ $n\mathscr{F}(Y) = 0$. Let $k \subset K$ be an extension of algebraically closed fields. Then the natural map $Spec\, K \xrightarrow{\pi} Spec\, k$ induces an isomorphism $\pi^* : \mathscr{F}(Spec\, k) \xrightarrow{\simeq} \mathscr{F}(Spec\, K)$.*

Remark 1.15. $Spec\, K$ is a pro-object in the category Sm/k. Moreover, schemes in the pro-system defining $Spec\, K$ may be chosen affine [5,18].

Proof. Here we mostly follow the strategy proposed in [5,18]. Let us take $\alpha \in \mathscr{F}(Spec\, k)$ such that $\pi^*(\alpha) = 0$. Since $\mathscr{F}(Spec\, K) = \lim_{\rightarrow} \mathscr{F}(U)$, we can choose a smooth affine variety $U \xrightarrow{\varphi} Spec\, k$ such that $\varphi^*(\alpha) = 0$. Since the field k is algebraically closed, there exists a k-point $Spec\, k \xrightarrow{x} U$. Since the composition $\mathscr{F}(Spec\, k) \xrightarrow{\varphi^*} \mathscr{F}(U) \xrightarrow{x^*} \mathscr{F}(Spec\, k)$ is the identity map, we obtain: $\alpha = 0$. This shows that π^* is a monomorphism.

Let us now prove the surjectivity of π^*. Suppose, we are given $\beta \in \mathscr{F}(Spec\, K)$. Choose an affine smooth variety U together with a morphism $Spec\, K \xrightarrow{y} U$ such that $\beta = y^*(\beta')$, where $\beta' \in \mathscr{F}(U)$. As before, U has to have a k-point $Spec\, k \xrightarrow{x} U$. Consider the commutative diagram

$$\begin{tikzcd}
U_K \arrow[r, "\bar{\pi}"] \arrow[d] & U \arrow[d] \\
Spec\, K \arrow[u, bend left, "\tilde{y}"] \arrow[u, bend right, "\tilde{x}"'] \arrow[ur, "y"] \arrow[r, "\pi"] & Spec\, k, \arrow[u, bend right, "x"']
\end{tikzcd} \tag{1.6}$$

where $U_K = Spec\, K \times_{Spec\, k} U$. Two K-points $x \circ \pi$ and y of U determine two K-points $\tilde{x}, \tilde{y}$ of U_K. We have: $\beta - \pi^* \circ x^*(\beta') = (y^* - \pi^* \circ x^*)(\beta') = (\tilde{y}^* - \tilde{x}^*) \circ \bar{\pi}^*(\beta') = 0$, because $\tilde{y}^* - \tilde{x}^* = 0$, due to the Rigidity Theorem. Thus, $\beta \in \operatorname{Im} \pi^*$. □

2. Orientable functors

The purpose of the current section is to introduce a class of "Orientable Functors" for which a family of transfers will be constructed in subsequent sections.

Let us consider a category Sm^2/k whose objects are pairs (X, U) such that $X \in Sm/k$ and U is open in X. Morphisms in this category are usual morphisms of pairs.

Orientable functors are special kinds of contravariant functors on this category. Typical examples are the functors, which are represented by commutative ring T-spectra and which are orientable in the sense of Morel [12]. Functors given by these examples are double-indexed and the product satisfies the usual skew-commutativity property. In order to get a notion of an orientable functor, we axiomatize these and some other properties of the above mentioned functors. In fact, we never really need double-indexing but we decided to keep it to be consistent with these typical examples.

Let us define by $R: Sm^2/k \to Sm^2/k$ the functor given by the rule: $R(X, U) = (U, \emptyset)$. Consider a functor (family of functors) $E^{p,q}: (Sm^2/k)^\circ \to \mathcal{A}b$ ($p, q \in \mathbb{Z}$) endowed with a family of natural transformations $\partial^{p,q}: E^{p,q} \circ R \to E^{p+1,q}$, satisfying the following list of axioms.

Remark 2.1. We will often write $E_Z^{*,*}(X)$ for $E^{*,*}(X, X-Z)$ and $E^{*,*}(X)$ for $E^{*,*}(X, \emptyset)$.

Eilenberg–Steenrod type axioms.

Axiom 2.2 (Localization). *Let* $(U, \emptyset) \xrightarrow{f} (X, \emptyset) \xrightarrow{j} (X, U)$ *be morphisms in* Sm^2/k *such that* j *is induced by* $X \xrightarrow{id} X$. *Then we have the following long exact sequence*:

$$\cdots \xrightarrow{j^*} E^{*,*}(X) \xrightarrow{f^*} E^{*,*}(U) \xrightarrow{\partial^{*,*}} E^{*+1,*}(X, U) \xrightarrow{j^*} \cdots$$

Axiom 2.3 (Excision). *Let* $X \overset{i}{\supseteq} X_0 \supseteq Z$, *where* X_0 *is open in* X *and* Z *is closed in* X. *Then the induced map* $i^*: E_Z^{*,*}(X) \xrightarrow{\simeq} E_Z^{*,*}(X_0)$ *is an isomorphism.*

Axiom 2.4 (Homotopy invariance). *The functor* $E^{*,*}$ *is homotopy invariant* (*see Definition* 1.9).

Homotopy purity.

Axiom 2.5. *Let* $B(X, Z) = (X \times \mathbb{A}^1)\hat{\ }_{Z \times \{0\}}$ *denote the deformation to the normal cone of the closed subscheme* Z *in* X. *Also, let* $\mathcal{N}$ *be the corresponding normal bundle* *over* Z *and* $i_0:\mathcal{N}\hookrightarrow B(X,Z)$ *and* $i_1:X\hookrightarrow B(X,Z)$ *be canonical embeddings over* 0 *and* 1, *respectively. In this case the induced maps*:

$$E_Z^{*,*}(\mathcal{N}) \xleftarrow{i_0^*} E_{Z\times\mathbb{A}^1}^{*,*}(B(X,Z)) \xrightarrow{i_1^*} E_Z^{*,*}(X)$$

are isomorphisms. (*For more details about the meaning of this axiom see Section* 4 *below and* [14, *Section* 3.2.3]).

Product structure.

Axiom 2.6 (Multiplicativity)**.** *We are given a pairing*

$$E^{p,q}(X)\otimes E_Z^{p',q'}(X)\xrightarrow{\smile} E_Z^{p+p',q+q'}(X),$$

which satisfies the following conditions:

(a) *functoriality*: *for elements* $\alpha\in E^{p,q}(X)$, $\beta\in E_Z^{p',q'}(X)$, *and a projective morphism* $f:Y\to X$, *one has*: $f^*(\alpha\smile\beta)=f^*(\alpha)\smile f^*(\beta)\in E_{f^{-1}(Z)}^{p+p',q+q'}(Y)$;

(b) *for every* X *there exists two-sided unit* $1_X\in E^{0,0}(X)$ *and it is functorial*;

(c) *associativity* (*provided that both sides are well defined*);

(d) *skew-commutativity*: *for* $\alpha\in E^{p,q}(X)$ *and* $\beta\in E^{p',q'}(X)$, *one has*: $\alpha\smile\beta=(-1)^{p\cdot p'}\beta\smile\alpha$.

Chern classes.

Axiom 2.7 (First Chern class)**.** *For every line bundle* L *over a smooth variety* X *we can choose a class* $c_1(L)\in E^{2,1}(X)$ (*called the first Chern class of the bundle* L) *such that the family* $c_1(-)$ *satisfies the following properties*:

(a) *If* $L\simeq L'$ *then* $c_1(L)=c_1(L')$;

(b) *For a morphism* $f:X\to Y$, *we have*: $f^*(c_1(L))=c_1(f^*(L))$;

(c) $c_1(\mathbf{1})=0$ (*Here and later on we denote by* $\mathbf{1}$ *the trivial line bundle. The nontriviality of the Chern class follows from the next axiom.*)

Axiom 2.8 (Projective Bundle Theorem)**.** *Let* $\mathscr{E}$ *be a vector bundle of rank* r *over* X. *Denote by* $\mathbb{P}(\mathscr{E})\xrightarrow{p}X$ *the projective bundle over* X *associated to* $\mathscr{E}$. *A fiber of this bundle over a point* $\{x\}$ *in* X *is the projective space of lines in the fiber* $\mathscr{E}_x$. *Let* $\mathscr{O}(-1)$ *be the tautological line bundle over* $\mathbb{P}(\mathscr{E})$ *and* $\xi=c_1(\mathscr{O}(-1))$ *be its first Chern class.*

The ring map $\varphi:E^{*,*}(X)[T]/(T^r)\to E^{*,*}(\mathbb{P}(\mathscr{E}))$ *such that* $\varphi|_{E^{*,*}(X)}=p^*$ *and* $\varphi(T)=\xi$ *is an isomorphism of graded* $E^{*,*}(X)$-*modules. In other words*, $E^{*,*}(\mathbb{P}(\mathscr{E}))$ *is a free* $E^{*,*}(X)$-*module with a base* $1,\xi,\xi^2,\dots,\xi^{r-1}$.

Moreover, *if the bundle* $\mathscr{E}$ *is trivial*, *the map* φ *is a ring isomorphism.*

Remark 2.9. For the class of functors representable by T-spectra (see Section 6.3) the previous Axiom may be given in a more simple way:

Axiom 2.8′. *Let* $\sigma \in E^{2,1}_{\{0\}}(\mathbb{A}^1)$ *be obtained by applying* T*-suspension to* $1 \in E^{0,0}(pt)$. *Denote by* $\bar{\sigma}$ *the image of* σ *under the map*: $E^{2,1}_{\{0\}}(\mathbb{A}^1) = E^{2,1}_{\{0\}}(\mathbb{P}^1) \hookrightarrow E^{2,1}(\mathbb{P}^1)$, *then*: $c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\bar{\sigma}$.

The equivalence of Axioms 2.8 and 2.8′ was stated by Morel and Suslin (see, for example [12]). We actually never use form 2.8′ except in Section 6.5, where the case of Algebraic Cobordism is treated.

The latter axiom enables us to apply the approach to the definition of higher Chern classes proposed by Grothendieck [6] (see also [12]).

Definition 2.10. We call the coefficients $c_i(\mathscr{E}) \in E^{2i,i}(X)$ in the relation

$$\xi^r - c_1(\mathscr{E})\xi^{r-1} + \cdots + (-1)^{r-1}c_{r-1}(\mathscr{E})\xi + (-1)^r c_r(\mathscr{E}) = 0$$

the Chern classes of the vector bundle $\mathscr{E}$. In particular, the last statement of Axiom 2.8 implies that all Chern classes (but $c_0 = 1$) of a trivial vector bundle are 0.

Corollary 2.11 (Of the definition). *Let* $f : Y \to X$ *be a morphism of smooth varieties. Let* $\mathscr{E}$ *be a vector bundle of dimension* r *over* X. *Then, for any* n *we have*

$$c_n(f^*\mathscr{E}) = f^*(c_n(\mathscr{E})).$$

Proof. *Let* ξ_X (resp. ξ_Y) be the first Chern class of the bundle $\mathcal{O}(-1)$ over $\mathbb{P}_X(\mathscr{E})$ (resp. $\mathbb{P}_Y(f^*\mathscr{E})$). Since $\xi_Y = f^*\xi_X$, we have a relation

$$\xi_Y^r + f^*(-c_1(\mathscr{E}))\xi_Y^{r-1} + \cdots + f^*((-1)^r c_r(\mathscr{E})) = 0 \qquad (2.1)$$

which holds in $E^{*,*}(\mathbb{P}_Y(f^*\mathscr{E}))$.

On the other hand, one has a similar relation for the class ξ_Y^r with coefficients $(-1)^i c_i(f^*(\mathscr{E}))$. Since $E^{*,*}(\mathbb{P}_Y(f^*\mathscr{E}))$ is a free $E^{*,*}(Y)$-module with a base $1, \xi_Y, \ldots, \xi_Y^{r-1}$, this implies the coincidence of these two relations and, therefore, the functoriality of higher Chern classes. □

The following two properties can be easily derived from Excision and Localization axioms.

Lemma 2.12 (Vanishing property). *For any X, we have*: $E^{*,*}_{\emptyset}(X) = 0$.

Lemma 2.13 (Finite additivity). $E^{*,*}(X_1 \sqcup X_2) \xrightarrow{\simeq} E^{*,*}(X_1) \oplus E^{*,*}(X_2)$.

Definition 2.14. We call a family of functors $E^{*,*} : (Sm^2/k)^{\circ} \to \mathscr{A}b$, satisfying Axioms 2.3–2.8 an orientable functor. The family of functors is called oriented if a collection of first Chern classes is specified.

Consider a functor $\mathfrak{G} : Sm/k \to Sm^2/k$ given as follows: $\mathfrak{G}(X) = (X, \emptyset)$. This functor embeds Sm/k into Sm^2/k as a full subcategory. Given a functor $E^{*,*} : (Sm^2/k)^{\circ} \to \mathscr{A}b$, we can consider the composite: $E^{*,*}\mathfrak{G}^{\circ} : (Sm/k)^{\circ} \to \mathscr{A}b$.

Theorem 2.15. *Let $E^{*,*}:(Sm^2/k)^{\circ} \to \mathscr{A}b$ be an oriented functor. Then for every projective morphism $f:X \to Y$ and its decomposition $X \overset{\tau}{\hookrightarrow} Y\times\mathbb{P}^n \overset{p^Y}{\to} Y$ we can canonically construct the push-forward map $f_*^{\tau}:E^{*,*}(X) \to E^{*+2r,*+r}(Y)$ $(r = dim\, Y - dim\, X)$ such that the functor $E^{*,*}\mathfrak{G}^{\circ}$ becomes a homotopy invariant functor with weak transfers for the class of all projective morphisms.*

The proof of the theorem is postponed till Section 5 when we develop all the necessary machinery.

We now present a method for constructing new orientable functors from a given orientable functor. Let us fix a variety $Y \in Sm/k$ and consider the functor $\mathfrak{Y}:Sm^2/k \to Sm^2/k$ defined as follows: $\mathfrak{Y}(X,U) = (Y\times X, Y\times U)$ for $(X,U)\in Ob(Sm^2/k)$ and $\mathfrak{Y}(f) = id_Y\times f$ for $f \in Mor(Sm^2/k)$. Let $E^{*,*}:(Sm^2/k)^{\circ} \to \mathscr{A}b$ be a family of functors. The composite $E^{*,*}\circ\mathfrak{Y}^{\circ}:(Sm^2/k)^{\circ} \to \mathscr{A}b$ gives another family of contravariant functors on the category Sm^2/k, which we call ${}^{Y}E^{*,*}$.

Proposition 2.16. *Let $E^{*,*}:(Sm^2/k)^{\circ} \to \mathscr{A}b$ be an orientable functor. Then for any smooth variety Y the family ${}^{Y}E^{*,*}$ is an orientable functor as well.*

Proof. Observe, first, that the inverse image map $\mathfrak{y}^*:E^{*,*}(-) \to E^{*,*}(Y\times -)$ defines a natural transformation of functors $\mathfrak{y}:E \to^{Y} E$. Eilenberg–Steenrod type Axioms 2.2–2.4 follow immediately from the definition of this natural transformation. The Homotopy purity axiom for the pair (X,Z) and for the functor ${}^{Y}E^{*,*}$ is equivalent to the Homotopy purity for the pair $(Y\times X, Y\times Z)$ and for the initial orientable functor $E^{*,*}$. The ring structure in ${}^{Y}E^{*,*}$ is obviously inherited from one in $E^{*,*}$. Finally, we define the first Chern class for a line bundle L over X as ${}^{Y}E^{2,1}(X) \ni \tilde{c}_1(L) = \mathfrak{y}^*(c_1(L))$. After this definition the Projective Bundle Theorem also follows immediately. □

Theorem 2.17. *Let $k \subset K$ be an extension of algebraically closed fields. Let also $E^{*,*}$ be an orientable functor vanishing after multiplication by n mutually prime to $Char\, k$. Then, for any $Y \in Sm/k$, we have:*

$$E^{*,*}(Y) \overset{\simeq}{\to} E^{*,*}(Y_K).$$

Proof. Proposition 2.16 and Theorem 2.15 show that the functor ${}^{Y}E^{*,*}$ is a homotopy invariant functor with weak transfers for the class of all projective morphisms (and, a fortiori, for the class of finite projective morphisms). Applying Theorem 1.14 to this functor, we get:

$$E^{*,*}(Y) = {}^{Y}E^{*,*}(k) \overset{\simeq}{\to} {}^{Y}E^{*,*}(K) = E^{*,*}(Y_K). \qquad \square \tag{2.2}$$

3. Construction of the Thom class

From now on $E^{*,*}$ denotes an orientable functor from the category $(Sm^2/k)^{\circ}$ to $\mathscr{A}b$.

Let X be a smooth variety over k and let $\mathscr{E}$ be a vector bundle of a constant rank r over X. Let $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E} \oplus \mathbf{1})$ be projectivizations of bundles $\mathscr{E}$ and $\mathscr{E} \oplus \mathbf{1}$, respectively. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[d, equal] & \mathscr{E} \arrow[d, equal] & & & \\
\mathbb{P}(\mathbf{1}) \arrow[r, hook, "j_1"] \arrow[rr, bend right, "s_1"'] & \mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathscr{E}) \arrow[r, hook, "i_1"] & \mathbb{P}(\mathscr{E} \oplus \mathbf{1}) & \mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathbf{1}) \arrow[l, hook', "i_{\mathscr{E}}"'] & \mathbb{P}(\mathscr{E}), \arrow[l, hook', "j_{\mathscr{E}}"'] \arrow[ll, bend right, "s_{\mathscr{E}}"']
\end{tikzcd} \tag{3.1}$$

where all arrows are corresponding canonical embeddings.

Lemma 3.1. *The following induced maps are isomorphisms*:
(a) $j_{\mathscr{E}}^* : E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathscr{E}))$;
(b) $i_1^* : E^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \to E^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathscr{E})) = E^{*,*}_X(\mathscr{E})$.

Proof. (a) Observe that $\mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathbf{1})$ is the total space of the tautological line bundle $\mathscr{O}(-1)$ over $\mathbb{P}(\mathscr{E})$ and the map $j_{\mathscr{E}}$ is its zero-section. The claim now follows from this observation and Proposition 1.10.

(b) It follows from the excision property. □

Lemma 3.2. *The following short sequences are exact*:
(a) $0 \to E^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{\varphi} E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{s_{\mathscr{E}}^*} E^{*,*}(\mathbb{P}(\mathscr{E})) \to 0$;
(b) $0 \to E^{*,*}_{\mathbb{P}(\mathscr{E})}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{s_1^*} E^{*,*}(\mathbb{P}(\mathbf{1})) \to 0$;
(c) $0 \to E^{*,*}_X(\mathscr{E}) \xrightarrow{\varphi} E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{s_{\mathscr{E}}^*} E^{*,*}(\mathbb{P}(\mathscr{E})) \to 0$.

Proof. We prove only assertion (a). Assertion (b) may be proven in the same way and, finally, (c) is just the application of Lemma 3.1(b) to (a). Let us look at the following fragment of the localization long exact sequence corresponding to the open embedding $\mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathbf{1}) \subset \mathbb{P}(\mathscr{E} \oplus \mathbf{1})$:

$$\cdots \to E^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1}) \backslash \mathbb{P}(\mathbf{1})) \to \cdots \tag{3.2}$$

Applying the statements of Lemma 3.1 to this sequence, we get

$$\cdots \to E^{*,*}_X(\mathscr{E}) \to E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{s_{\mathscr{E}}^*} E^{*,*}(\mathbb{P}(\mathscr{E})) \to \cdots \tag{3.3}$$

To complete the proof, it is sufficient to show that the map

$$E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1})) \xrightarrow{s_{\mathscr{E}}^*} E^{*,*}(\mathbb{P}(\mathscr{E})) \tag{3.4}$$

is an epimorphism. The Projective Bundle Theorem tells us that both $E^{*,*}(X)$-algebras $E^{*,*}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1}))$ and $E^{*,*}(\mathbb{P}(\mathscr{E}))$ are generated by the first Chern classes of the corresponding tautological bundles. The map $s_{\mathscr{E}}^*$ is, obviously, an $E^{*,*}(X)$-module homomorphism. Finally, the restriction of the tautological line bundle on $\mathbb{P}(\mathscr{E} \oplus \mathbf{1})$ to $\mathbb{P}(\mathscr{E})$

coincides with the tautological line bundle on $\mathbb{P}(\mathscr{E})$. Since the first Chern class is functorial (Axiom 2.7(b)), the surjectivity follows. □

Let ζ and $\xi \in E^{2,1}(X)$ be the first Chern classes of the tautological line bundles over projectivizations $\mathbb{P}(\mathscr{E} \oplus \mathbf{1})$ and $\mathbb{P}(\mathscr{E})$, respectively. As before, the functoriality of the first Chern class implies that $s_{\mathscr{E}}^*(\zeta) = \xi$. Consider now the Chern polynomial

$$c_t(\mathscr{E}) = \sum_{k=0}^{r} (-1)^{r-k} c_{r-k}(\mathscr{E}) t^k. \tag{3.5}$$

We have: $c_\zeta(\mathscr{E}) \in E^{2r,r}(\mathbb{P}(\mathscr{E} \oplus \mathbf{1}))$. It follows from the Projective Bundle Theorem, that $s_{\mathscr{E}}^*(c_\zeta(\mathscr{E})) = c_{s_{\mathscr{E}}^*\zeta}(\mathscr{E}) = c_\xi(\mathscr{E}) = 0$. So that, from Lemma 3.2(c), we have $c_\zeta(\mathscr{E}) \in \operatorname{Im} \varphi(E_X^{2r,r}(\mathscr{E}))$.

Definition 3.3 (*The Thom class*)**.** We set $th(\mathscr{E}) = \varphi^{-1} c_\zeta(\mathscr{E}) \in E_X^{2r,r}(\mathscr{E})$ and call $th(\mathscr{E})$ the Thom class of the vector bundle $\mathscr{E}$.

Proposition 3.4. *Let $f: Y \to X$ be a morphism of smooth varieties. Let $\mathscr{E}$ be a vector bundle of dimension r over X. Then, we have*

$$th(f^*\mathscr{E}) = f^*(th(\mathscr{E})).$$

Proof. Corollary 2.11 implies that the corresponding Chern polynomials coincide:

$$f^*(c_t(\mathscr{E})) = c_t(f^*\mathscr{E}). \tag{3.6}$$

Since the inverse image of the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathscr{E}\oplus\mathbf{1})}(-1)$ equals to $\mathcal{O}_{\mathbb{P}_Y(f^*\mathscr{E}\oplus\mathbf{1})}(-1)$, we get the desired result. □

4. Gysin map for closed embedding

Let $\tau: X \hookrightarrow Y$ be a closed embedding of codimension d of smooth k-varieties and let $\mathscr{N} = \mathscr{N}_{Y/X}$ be the corresponding normal bundle. We need to recall a well-known construction of deformation to the normal cone. Let $B = B(Y,X)$ denote the blow-up of $Y \times \mathbb{A}^1$ with center at $X \times \{0\}$. Considering the fibers of the map $B(Y,X) \to Y \times \mathbb{A}^1 \to \mathbb{A}^1$ over the points $\{0\}$ and $\{1\}$ one easily obtains two embeddings: $i_0: \mathbb{P}(\mathscr{N} \oplus \mathbf{1}) \hookrightarrow B(Y,X)$ and $i_1: Y \hookrightarrow B(Y,X)$. The subvariety $X \times \mathbb{A}^1$ contains the center $X \times \{0\}$ of the blow-up as a divisor. Therefore, it lifts canonically to a subvariety in $B(Y,X)$. Since $X \times \mathbb{A}^1$ crosses $\mathbb{P}(\mathscr{N} \oplus \mathbf{1})$ along $\mathbb{P}(\mathbf{1})$ and crosses $i_1(Y)$ along $i_1(X)$, one has following embeddings of the pairs:

$$(\mathscr{N}, \mathscr{N} - X) \overset{i_0}{\hookrightarrow} (B, B - X \times \mathbb{A}^1) \overset{i_1}{\hookleftarrow} (Y, Y - X). \tag{4.1}$$

Axiom 2.5 is read as

$$E_X^{*,*}(\mathscr{N}) \underset{\simeq}{\overset{i_0^*}{\longleftarrow}} E_{X \times \mathbb{A}^1}^{*,*}(B) \underset{\simeq}{\overset{i_1^*}{\longrightarrow}} E_X^{*,*}(Y). \tag{4.2}$$

Construction-Defintion 4.1. We define the Gysin map τ_* as the following chain of maps:

$$E^{*-2d,*-d}(X) \xrightarrow{\smile th(\mathcal{N})} E^{*,*}_X(\mathcal{N}) \xrightarrow{i_1^*\circ(i_0^*)^{-1}} E^{*,*}_X(Y) \xrightarrow{j^*} E^{*,*}(Y),$$

where $j:(Y,\emptyset)\hookrightarrow(Y,Y-X)$ and $\smile th(\mathcal{N})$ denotes the composition of the canonical isomorphism $(s^*)^{-1}:E^{*,*}(X)\xrightarrow{\simeq}E^{*,*}(\mathcal{N})$ induced by the zero-section of the bundle $\mathcal{N}$ and the multiplication with the Thom class $th(\mathcal{N})$.

Let us prove some properties of the constructed Gysin map.

Proposition 4.2 (Base change property). *For a transversal square*

$$\begin{tikzcd} X' \arrow[r, hook, "\tau'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "\tau"] & Y \end{tikzcd}$$

the square

$$\begin{tikzcd} E^{*,*}(X') \arrow[r, "\tau'_*"] & E^{*,*}(Y') \\ E^{*,*}(X) \arrow[r, "\tau_*"] \arrow[u, "g^*"] & E^{*,*}(Y) \arrow[u, "f^*"] \end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd} E^{*-2d,*-d}(X') \arrow[r, "\smile th(\mathcal{N}')"] & E^{*,*}_{X'}(\mathcal{N}') & E^{*,*}_{X'\times\mathbb{A}^1}(B') \arrow[l, "i_0'^*"'] \arrow[r, "j'^*i_1'^*"] & E^{*,*}(Y') \\ E^{*-2d,*-d}(X) \arrow[r, "\smile th(\mathcal{N})"] \arrow[u, "g^*"] & E^{*,*}_X(\mathcal{N}) \arrow[u, "N(g)^*"] & E^{*,*}_{X\times\mathbb{A}^1}(B) \arrow[l, "i_0^*"'] \arrow[r, "j^*i_1^*"] \arrow[u, "B(f)^*"] & E^{*,*}(Y). \arrow[u, "f^*"] \end{tikzcd} \qquad (4.3)$$

All the squares but the very left one are obviously commutative. Therefore, to obtain the relation $\tau'_*g^*=f^*\tau_*$, we just need to check that $g^*(th(\mathcal{N}))=th(g^*(\mathcal{N}))=th(\mathcal{N}')$, which holds due to Proposition 3.4, since the initial square is transversal. □

Proposition 4.3 (Finite additivity). *Let us consider the commutative diagram*

$$\begin{tikzcd} Z_0 \arrow[dr, "j_0"] \arrow[ddr, bend right, "\tau_0"'] & & Z_1 \arrow[dl, "j_1"'] \arrow[ddl, bend left, "\tau_1"] \\ & Z=Z_0\coprod Z_1 \arrow[d, hook, "\tau"] & \\ & X & \end{tikzcd}$$

with canonical maps j_0, j_1 such that the map τ is a closed embedding. Then

$$\tau_* = \tau_{0,*} j_0^* + \tau_{1,*} j_1^*.$$

Proof. Consider the commutative diagrams of natural inclusions (for $m = 0, 1$)

$$\begin{tikzcd} (Z_m, \emptyset) \arrow[r, hook, "j_m"] \arrow[rr, hook, bend right, "\psi_m"'] & (Z, \emptyset) \arrow[r, hook, "\varphi_m"] & (Z, Z_{1-m}) \end{tikzcd} \tag{4.4}$$

Applying the functor $E^{*,*}$ to both diagrams and adding the results, we obtain the following commutative diagram:

$$\begin{tikzcd} E^{*,*}(Z) \arrow[r, "{(j_0^*, j_1^*)}", "\simeq"'] & E^{*,*}(Z_0) \oplus E^{*,*}(Z_1) \\ E^{*,*}_{Z_0}(Z) \oplus E^{*,*}_{Z_1}(Z) \arrow[u, "\varphi_0^* + \varphi_1^*"] \arrow[ur, "\simeq", "\psi_0^* \oplus \psi_1^*"'] & \end{tikzcd} \tag{4.5}$$

In the diagram above, the horizontal arrow is an isomorphism since the functor $E^{*,*}$ is additive by 2.13. The excision property 2.3 makes the diagonal arrow an isomorphism as well. Since two arrows in the considered diagram are isomorphisms, the last one $\varphi_0^* + \varphi_1^*$ is also an isomorphism. One can easily see that the sum: $\varphi_0^*[\psi_0^*]^{-1} j_0^* + \varphi_1^*[\psi_1^*]^{-1} j_1^*$ defines the identity map on $E^{*,*}(Z)$. Let us now rewrite what we want to prove: $\tau_{0,*} j_0^* + \tau_{1,*} j_1^* = \tau_* = \tau_* \circ [\varphi_0^*[\psi_0^*]^{-1} j_0^* + \varphi_1^*[\psi_1^*]^{-1} j_1^*]$. Therefore, in order to prove the proposition, it is sufficient to check that $\tau_{m,*} = \tau_* \varphi_m^* [\psi_m^*]^{-1}$ for $m = 0, 1$. To verify this relation for $m = 1$ (the case of $m = 0$ is exactly the same), consider the diagram

$$\begin{tikzcd} (\mathcal{N}, \mathcal{N} - Z) \arrow[r] \arrow[d] & (B, B - Z \times \mathbb{A}^1) \arrow[d] & (X, X - Z) \arrow[l] \arrow[d] \\ (\mathcal{N}, \mathcal{N} - Z_1) \arrow[r] & (B, B - Z_1 \times \mathbb{A}^1) \arrow[d, "\sigma"] & (X, X - Z_1) \arrow[l] \arrow[d, equal] \\ (\mathcal{N}_1, \mathcal{N}_1 - Z_1) \arrow[r] \arrow[u] & (B_1, B_1 - Z_1 \times \mathbb{A}^1) & (X, X - Z_1) \arrow[l] \end{tikzcd} \tag{4.6}$$

The top (bottom) row of this diagram corresponds to the morphism $Z \overset{\tau}{\hookrightarrow} X$ ($Z_1 \overset{\tau_1}{\hookrightarrow} X$, respectively). We also put: $B = B(X, Z)$, $B_1 = B(X, Z_1)$, and $\mathcal{N}_1 = \mathcal{N}|Z_1$. The morphism σ is the blow-down of the component Z_0.

Since $\sigma^{-1}(Z_1 \times \mathbb{A}^1) = Z_1 \times \mathbb{A}^1$, the morphism σ gives us a well-defined morphism of pairs: $(B, B - Z_1 \times \mathbb{A}^1) \overset{\sigma}{\to} (B_1, B_1 - Z_1 \times \mathbb{A}^1)$.

Applying the functor $E^{*,*}$ to Diagram 4.6 and combining the result with the definition of Gysin map, we obtain the following diagram (here and below we sometimes omit the indexes for the brevity):

$$
\begin{tikzcd}
E(Z) \arrow[r, "\smile th(\mathcal{N})"] \arrow[rrrr, bend left, "\tau_*"] & E_Z(\mathcal{N}) & E_{Z\times\mathbb{A}^1}(B) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & E_Z(X) \arrow[r] & E(X) \\
E_{Z_1}(Z) \arrow[u, "\varphi_1^*"] \arrow[r, "\smile th(\mathcal{N})"] \arrow[d, "\psi_1^*"', "\simeq"] & E_{Z_1}(\mathcal{N}) \arrow[u] \arrow[d, "N(\psi_1)^*"', "\simeq"] & E_{Z_1\times\mathbb{A}^1}(B) \arrow[l] \arrow[r] \arrow[u] & E_{Z_1}(X) \arrow[r] \arrow[u] & E(X) \arrow[u, equal] \\
E(Z_1) \arrow[r, "\smile th(\mathcal{N}_1)"] \arrow[rrrr, bend right, "\tau_{1,*}"'] & E_{Z_1}(\mathcal{N}_1) & E_{Z_1\times\mathbb{A}^1}(B_1) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u, "\sigma^*"'] & E_{Z_1}(X) \arrow[r] \arrow[u, equal] & E(X). \arrow[u, equal]
\end{tikzcd}
\tag{4.7}
$$

One can easily verify that this diagram commutes and implies the desired relation. □

5. Transfer map and the proof of Theorem 2.15

Now we construct the family of transfer maps. Let $f: X \to Y$ be a projective morphism of codimension d, with a decomposition

$$
\begin{tikzcd}
X \arrow[r, hook, "\tau"] \arrow[rr, bend right, "f"'] & Y\times\mathbb{P}^n \arrow[r, "p^Y"] & Y,
\end{tikzcd}
\tag{5.1}
$$

where τ is a closed embedding and p^Y denotes the projection.

Definition 5.1. Define a map $p^Y_*: E^{*,*}(Y\times\mathbb{P}^n) \to E^{*-2n,*-n}(Y)$ as follows. By the Projective Bundle Theorem one has a direct sum decomposition

$$E^{*,*}(Y\times\mathbb{P}^n) = \bigoplus_{i=0}^{n} E^{*-2i,*-i}(Y)\times\xi^i, \tag{5.2}$$

where $\xi = c_1(\mathcal{O}(-1))$. Set p^Y_* be projection on the summand $E^{*-2n,*-n}(Y)$.

Proposition 5.2. *The following properties hold for the map* p^Y_*:

(a) *base change property for an arbitrary morphism*;

(b) $p^Y_*(\xi^n) = 1$.

Proof. Both properties are direct consequences of the definition above. □

Definition 5.3. For every diagram of the form 5.1 we define the transfer map f^τ_*: $E^{*,*}(X) \to E^{*+2d,*+d}(Y)$ by setting $f^\tau_* = p^Y_* \circ \tau_*$, where $d = dim\, Y - dim\, X$.

To complete the proof of Theorem 2.15 we only need to check that Properties 1.5–1.7 hold for the constructed transfer maps. We shall do this in Propositions 5.4–5.6.

Proposition 5.4 (Compare with 1.5). *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a Cartesian square where the projective morphism $X \xrightarrow{f} Y$ is *decomposed as in* 5.1 *and the square*

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, hook, "\tau'"] \arrow[d, "g'"] & Y' \times \mathbb{P}^n \arrow[d, "G = g \times id"] \\ X \arrow[r, hook, "\tau"] & Y \times \mathbb{P}^n \end{tikzcd} \tag{5.3}$$

is transversal. Then the diagram

$$\begin{tikzcd} E^{*,*}(X') \arrow[r, "f'_*"] & E^{*,*}(Y') \\ E^{*,*}(X) \arrow[r, "f_*"] \arrow[u, "g'^*"] & E^{*,*}(Y). \arrow[u, "g^*"] \end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd} & Y' \times \mathbb{P}^n \arrow[rr, "G"] \arrow[ddl, "p^{Y'}"] & & Y \times \mathbb{P}^n \arrow[ddl, "p^Y"] \\ X' \arrow[ur, hook, "\tau'"] \arrow[rr, "g'"] \arrow[d, "f'"] & & X \arrow[ur, hook, "\tau"] \arrow[d, "f"] & \\ Y' \arrow[rr, "g"] & & Y & \end{tikzcd} \tag{5.4}$$

where $G = g \times id$. Since square (5.3) is transversal, Proposition 4.2 implies that: $G^* \tau_* = \tau'_* g'^*$. On the other hand, from Proposition 5.2, we have: $g^* p^Y_* = p^{Y'}_* G^*$. Therefore, one gets: $g^* f_* = g^* p^Y_* \tau_* = p^{Y'}_* G^* \tau_* = p^{Y'}_* \tau'_* g'^* = (f')_* g'^*$. The base change property is proven. $\square$

Proposition 5.5 (Compare with 1.6). *Let* $X = X_0 \sqcup X_1$, $j_m : X_m \hookrightarrow X$ $(m = 0, 1)$ *be embedding maps, and* $f : X \to Y$. *Setting* $f j_m = f_m$, *one has*

$$f_{0,*} j_0^* + f_{1,*} j_1^* = f_*. \tag{5.5}$$

Proof. The Additivity Property 4.3 gives us the relation: $\tau_* = \tau_{0,*}j_0^* + \tau_{1,*}j_1^*$. So, we have the equality: $f_* = p_*^Y\tau_* = p_*^Y(\tau_{0,*}j_0^* + \tau_{1,*}j_1^*) = f_{0,*}j_0^* + f_{1,*}j_1^*$, which proves the additivity property. □

The last step is to check the normalization property.

Proposition 5.6 (Compare with 1.7). *Let $X = Y = Spec(k)$ and $f: X \to Y$ be the identity morphism. Then for every decomposition $pt \overset{\tau}{\hookrightarrow} \mathbb{P}^n \overset{p}{\to} pt$ the map $f_*^\tau: E^{*,*}(Spec\, k) \to E^{*,*}(Spec\, k)$ is the identity.*

Proof. Without loss of generality, we may assume that: $\tau(Spec(k)) = [0{:}\ 0{:}\ \cdots{:}\ 1]$. Granting for a moment the equality $\tau_*(1) = \xi^n$, one can easily see that the desired property follows from a chain of equalities: $f_*(1) = p_*\tau_*(1) = p_*(\xi^n) = 1$. The right-most equality follows from Proposition 5.2. Since both maps τ_* and p_* are $E^{*,*}(Spec\, k)$-linear, the normalization property follows. □

Lemma 5.7. $\tau_*(1) = \xi^n$, *where* $\xi = c_1(\mathcal{O}(-1)) \in E^{2,1}(\mathbb{P}^n)$.

Proof. Let $\mathcal{N}$ denote the normal bundle corresponding to the embedding $pt \overset{\tau}{\hookrightarrow} \mathbb{P}^n$. We have: $\mathcal{N} = \mathbf{1}^n$ and $\mathbb{P}^n = \mathbb{P}(\mathcal{N} \oplus \mathbf{1})$. Consider the square which is obviously transversal:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, hook] \arrow[d, equal] & \mathbb{A}^n \arrow[d, hook, "j"] \\ \mathrm{pt} \arrow[r, hook] & \mathbb{P}^n. \end{tikzcd} \tag{5.6}$$

Here j is the open embedding of $\mathbb{A}^n$ into $\mathbb{P}^n$, given by the formula: $j(x_1,\ldots,x_n) = (x_1{:}\ \cdots{:}\ x_n{:}\ 1)$. The corresponding base-change diagram for the Gysin map looks now as follows:

$$\begin{tikzcd} E(Spec\, k) \arrow[r, "\smile th(\mathcal{N})"] \arrow[d, equal] & E_{\{0\}}(\mathcal{N}) \arrow[d, equal] & E_{\{0\}\times\mathbb{A}^1}(B(\mathbb{P}^n,\{0\})) \arrow[l, "\bar{i}_0^*"'] \arrow[r, "\bar{i}_1^*"] \arrow[d] & E_{\{0\}}(\mathbb{P}^n) \arrow[r] \arrow[d, "\simeq"', "j^*"] & E(\mathbb{P}^n) \\ E(Spec\, k) \arrow[r, "\smile th(\mathcal{N})"] & E_{\{0\}}(\mathcal{N}) & E_{\{0\}\times\mathbb{A}^1}(B(\mathbb{A}^n,\{0\})) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] & E_{\{0\}}(\mathbb{A}^n). \end{tikzcd} \tag{5.7}$$

The map j^* in the diagram above is an isomorphism due to the excision axiom. Since the Chern classes of a trivial vector bundle vanish in positive dimensions, we have the equality $c_t(\mathcal{N}) = t^n$.

From the definition of the Thom class we can see that $j^*(c_\xi(\mathcal{N})) = th(\mathcal{N})$. Therefore, in order to verify the desired relation we just need to check that $i_1^*[i_0^*]^{-1} = id$. □

Lemma 5.8. *Let us consider the diagram*

$$E_{\{0\}}^{*,*}(\mathbb{A}^n) \overset{i_0^*}{\leftarrow} E_{\{0\}\times\mathbb{A}^1}^{*,*}(B(\mathbb{A}^n,\{0\})) \overset{i_1^*}{\to} E_{\{0\}}^{*,*}(\mathbb{A}^n),$$

where maps i_0^*, i_1^* are isomorphisms (see 2.5) induced by embeddings over the points $\{0\}$ and $\{1\}$, described above. Then $i_1^*[i_0^*]^{-1} = id$.

Proof. Observe first, that $B(\mathbb{A}^n,\{0\})=B=(\mathbb{A}^n\times\mathbb{A}^1)\widehat{\ }_{\{0\}}$ is the total space of the tautological line bundle over $\mathbb{P}^n$. Consider the following Cartesian square:

$$\begin{tikzcd} L \arrow[r, "J"] \arrow[d, "q"] & B \arrow[d, "\bar{q}"] \\ \mathbb{A}^n \arrow[r, hook, "j"] & \mathbb{P}^n. \end{tikzcd} \tag{5.8}$$

In this square $L=B\times_{\mathbb{P}^n}\mathbb{A}^n$ is the restriction of the vector bundle B to an open subset of the base. Evidently, the map $i_0:\mathcal{N}=\mathbb{A}^n\to B$ can be decomposed as $J\circ j_0$, where j_0 is the zero-section of the bundle L over $\mathbb{A}^n$.

In the same way the embedding i_1 comes from the section $j_1:\mathbb{A}^n\to L$ given by $x\mapsto(x,1)\times x\in L$.

The projection map $q:L\to\mathbb{A}^n$ clearly induces a morphism of pairs $(L,L-\{0\}\times\mathbb{A}^1)\to(\mathbb{A}^n,\mathbb{A}^n-\{0\})$, which we also denote by q. Since L is a vector bundle over $\mathbb{A}^n$, the map q^* is an isomorphism (inverse to j_0^*). Thus, we have: $j_1^*[j_0^*]^{-1}=j_1^*q^*=(qj_1)^*=id$. Since $i_k^*=j_k^*J^*$ $(k=0,1)$, the relation $i_1^*[i_0^*]^{-1}=id$ follows. $\square$

6. Applications

In this section we give a brief exposition of some important examples of orientable functors. By Theorem 2.15, all these examples are also homotopy invariant functors with transfers and, therefore, the rigidity property holds for all of them. Certainly, in order to satisfy the conditions of Theorem 1.13 all the considered theories should be taken with finite coefficients.

6.1. *Étale cohomology*

Let us set: $E_Z^{p,q}(X)=H_Z^p(X_{et},\mu_n^{\otimes q})$, where μ_n is the sheaf of nth roots of unit and $(n,Char\,k)=1$. All the needed properties of this functor are checked in [11]. Localization, excision, and homotopy invariance axioms are verified in Propositions III.1.25, III.1.27, and Corollary VI.4.20, respectively. The multiplicative structure is discussed in V.1. Finally, the Chern class theory is developed in Section VI.10.

6.2. *Algebraic K-theory*

In order to consider the K-theory in the context of orientable functors we need, first, to chose a model which is relative and functorial with respect to morphisms of pairs. For this end, we set $E_Z^{p,q}(X)=K_{2q-p}(X\,\mathrm{on}\,Z)$, where the groups on the right-hand side are Thomason–Trobaugh's K-groups (see [21]). The localization exact sequence is contained in [21, Theorem 5.1], the excision property follows from [21, 3.19]. Sections 4.1–4.12 of [21] are devoted to the proof of the projective bundle theorem.

For a smooth quasi-projective variety X and its closed subscheme Z it is shown that $K_*(X \operatorname{on} Z) = K'_*(Z)$, where the right-hand side groups are Quillen's groups of the category of coherent sheaves. Since these groups are homotopy invariant [16, Proposition 7.4.1], the homotopy invariance and homotopy purity axioms hold.

Starting from the pairing, given in [21, 3.15]:

$$K(X \operatorname{on} Y) \wedge K(X \operatorname{on} Z) \to K(X \operatorname{on} Y \cap Z), \tag{6.2.1}$$

one can easily construct the product on $E^{*,*}$ satisfying desired conditions.

Finally, in order to cook up the Chern classes, we just set

$$c_1(L) = [\mathbf{1}] - [L^{\vee}] \in K_0(X) \tag{6.2.2}$$

for a line bundle L over X.

6.3. Theories representable by T-spectra

Before proceeding with other two examples, let us make a short survey of cohomology theories represented by T-spectra. In such case Axioms 2.2–2.5 are actually built into the structure of spectrum and, therefore, for concrete examples we should verify only the existence of the multiplicative structure and Chern classes.

Discussing T-spectra we will use many pieces of notation without defining them. For all such categories, objects, and terminology given in *italic*, the reader should refer to papers [14,22,24,26,27].

A *T-spectrum* is a sequence of pointed *spaces* $E_i \in \mathbf{Spc}_*$ endowed with bonding maps $T \wedge E_i \to E_{i+1}$. (T is the *Tate object*, see [14, Section 3.2.2]). T-spectra form a homotopy category $\mathbf{SHot}^T(k)$ which happens to be equivalent to stable homotopy category $\mathbf{SHot}$ (see [26, Definition 2.1 and below]). There exists a functor $\Sigma_T^{\infty} : \mathbf{Spc}_* \to \mathbf{SHot}^T(k)$ corresponding to every space its suspension T-spectrum.

For a given T-spectrum $\mathbf{E}$ the corresponding cohomology theory is defined (see [24]) by setting

$$E^{p,q}(X) = Hom_{\mathbf{SHot}}(\Sigma_T^{\infty}(X_+), \mathbf{E}(q)[p]). \tag{6.3.1}$$

This construction has an evident extension to the relative cohomology, since the category $\mathbf{Spc}$ admits fibered coproducts:

$$E^{p,q}(X, U) = Hom_{\mathbf{SHot}}(\Sigma_T^{\infty}(X_+/U_+), \mathbf{E}(q)[p]). \tag{6.3.2}$$

Proposition 6.3.1. *Axioms* 2.2–2.5 *hold for any theory represented by a T-spectrum.*

Proof. Let us assume that $\mathcal{U}$ and $\mathcal{V}$ make an open Zariski covering of a variety X. Consider the following *distinguished square* in the category $\mathbf{Spc}$:

$$\begin{array}{ccc} \mathcal{U} \cap \mathcal{V} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & X. \end{array} \tag{6.3.3}$$

Since the top horizontal arrow in this square is a *cofibration*, we can apply [24, Proposition 4.11] and obtain the *distinguished triangle*

$$(\mathscr{U} \cap \mathscr{V})_+ \to \mathscr{U}_+ \oplus \mathscr{V}_+ \to X_+ \to (\mathscr{U} \cap \mathscr{V})_+[1]. \tag{6.3.4}$$

By basic properties of triangulated categories (see [8, Proposition 1.1(b)]), we obtain a Mayer–Vietoris long exact sequence

$$\cdots \to E^{p-1,q}(\mathscr{U} \cap \mathscr{V}) \to E^{p,q}(X) \to E^{p,q}(\mathscr{U}) \oplus E^{p,q}(\mathscr{V}) \to E^{p,q}(\mathscr{U} \cap \mathscr{V}) \to \cdots \tag{6.3.5}$$

In the same way one can get the localization long exact sequence, starting from the distinguished square

$$\begin{tikzcd} \mathscr{U} \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/\mathscr{U}. \end{tikzcd} \tag{6.3.6}$$

Let now $X \supseteq X_0 \supseteq Z$ be as in the Excision axiom. One can easily deduce the axiom from the Mayer–Vietoris sequence applied to the covering $\mathscr{U} = X_0$, $\mathscr{V} = X - Z$.

Finally, we should mention that the constructed theory is homotopy invariant by the definition of $\mathbf{SHot}^T(k)$. Concerning the Homotopy Purity axiom, we refer to [14, Proposition 3.2.24]. This proposition asserts that the maps i_0, i_1 from 2.5 are $\mathbb{A}^1$-weak equivalences and, therefore, induce isomorphisms in all theories represented by T-spectra. □

6.4. Motivic cohomology

For a smooth variety X over the field k we define its *motive* $M(X)$ as an object in the category $DM^-(k)$ given by the complex $C^*(\mathbb{Z}_{tr}(X))$ (see the definition just before [20, Theorem 1.5]).

We define motivic cohomology groups $H^{p,q}_{\mathscr{M}}(X)$ of X as the Zariski hypercohomology $H^p_{Zar}(X, \mathbb{Z}(q))$, where $\mathbb{Z}(n) = M(\mathbb{G}_m^{\wedge n})[-n]$ (see [20, Definition 3.1]).

Corollary 1.2 and Theorem 1.5 of [20] show that

$$H^{p,q}_{\mathscr{M}}(X) = H^p_{Nis}(X, \mathbb{Z}(q)) = Hom_{DM^-(k)}(M(X), \mathbb{Z}(q)[p]). \tag{6.4.1}$$

For a smooth variety X and its closed subset Z we define a motive with support $M_Z(X)$ as $C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X-Z))$ (see the proof of [20, Lemma 4.11]).

Let us now define $H^{p,q}_{\mathscr{M};Z}(X)$ as the group $Hom_{DM^-}(M_Z(X), \mathbb{Z}(q)[p])$ (see [20, Theorem 1.5] and the proof of [20, Corollary 8.4]).

In order to check most of the axioms, we describe a T-spectrum, which represents motivic cohomology. This spectrum is an algebraic analogue of the Eilenberg–Mac Lane spectrum in topology which is associated to a coefficient ring R and is denoted by $\mathbf{H}_R$ (see [24, Section 6.1] for the definition). The motivic cohomology of the smooth variety X over a field k admitting resolution of singularities is

$$H^{p,q}_{\mathscr{M}}(X,R) = Hom_{\mathbf{SHot}}(\Sigma^{\infty}_T(X_+), \mathbf{H}_R(q)[p]). \tag{6.4.2}$$

It is mentioned in [24, Section 6.1] that this construction gives the same cohomology groups as ones defined before. Since motivic cohomology is representable by a T-spectrum, we can just apply the result of the previous subsection to verify Eilenberg–Steenrod type axioms.

Remark 6.4.1. All these axioms can be verified using just the first definition by manipulating with exact triangles in the category of motives.

Now we define the cup-products as it is required by Axiom 2.6. Consider, first, the short exact sequence of sheaves with transfers:

$$0 \to \mathbb{Z}_{tr}(X - Z) \to \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X - Z) \to 0. \tag{6.4.3}$$

Since the *tensor product* $\otimes_{tr}$ with the sheaf $\mathbb{Z}_{tr}(X)$ is a right exact functor on the category of *Nisnevich sheaves with transfers* and $\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$ (see [20, the text just above Lemma 2.1]), one has a canonical isomorphism of *sheaves with transfers*

$$\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X - Z) = \mathbb{Z}_{tr}(X \times X)/\mathbb{Z}_{tr}(X \times X - Z). \tag{6.4.4}$$

Applying the functor C^* one gets a canonical isomorphism of motives $M_{X\times Z}(X\times X) = M(X) \otimes M_Z(X)$. We define the cup-product of $a \in H^{p,q}_{\mathscr{M}}(X)$ and $b \in H^{r,s}_{\mathscr{M};Z}(X)$ as the composite morphism

$$\begin{tikzcd}
M_Z(X) \arrow[rrr, "\smile"] \arrow[d, "\Delta_*"'] & & & \mathbb{Z}_{tr}(q+s)[p+r] \arrow[d, equal] \\
M_{X\times Z}(X\times X) \arrow[r, equal] & M(X)\otimes M_Z(X) \arrow[rr, "a\otimes b"] & & \mathbb{Z}_{tr}(q)[p]\otimes\mathbb{Z}_{tr}(s)[r],
\end{tikzcd} \tag{6.4.5}$$

where the morphism Δ_* is induced by the diagonal map and the right vertical identification is established in [20, Lemma 3.2].

It remains to define Chern classes. By [20, Corollary 3.2.1] the group $H^{2,1}_{\mathscr{M}}(X)$ of a smooth variety X is canonically isomorphic to Picard group $Pic(X)$. We set $c_1(L) = [L] \in Pic(X)$ for a line bundle L over X. Obviously, Axiom 2.7 holds for this Chern class. Finally, [20, Theorem 4.5] computes the motive of the projectivization of n-dimensional vector bundle $\mathbb{P}(\mathscr{E}) \to X$ in terms of the motive of X:

$$M(\mathbb{P}(\mathscr{E})) = M(X) \oplus M(X)(1)[2] \oplus \cdots \oplus M(X)(n)[2n] \tag{6.4.6}$$

and [20, Theorem 4.12] finishes the proof of the Projective Bundle Theorem. The main result of [25] allows us to get rid of the “resolution of singularities” assumption in [20, 4.12].

6.5. Algebraic cobordism

We start recalling the construction of the spectrum $\mathbf{MGL} = (MGL(n), c_n)$ from [24, Section 6.3]. Let us denote by $Gr(n,N)$ the Grassmannian variety of n-planes in $\mathbb{A}^N$. The canonical embeddings $\mathbb{A}^N \hookrightarrow \mathbb{A}^{N+1}$ induce the maps $Gr(n,N) \hookrightarrow Gr(n,N+1)$ and we denote $colim_N Gr(n,N)$ by $Gr(n)$. Let $\gamma_{n,N}$ be a tautological vector bundle over $Gr(n,N)$. Following [24], we define the Thom space of the vector bundle E over X as $Th(E) \overset{def}{=} E/(E - s(X))$, where $s : X \to E$ is the zero-section. The embeddings $Gr(n,N) \hookrightarrow Gr(n,N+1)$ induce morphisms $Th(\gamma_{n,N}) \to Th(\gamma_{n,N+1})$. We set $MGL(n) = colim_N Th(\gamma_{n,N})$.

Then, the embedding map $Gr(n,N) \overset{f_n}{\hookrightarrow} Gr(n+1, N+1)$, given as $f_n(x) = \mathbf{1} \oplus x$, induces the canonical splitting $f^*(\gamma_{n+1,N+1}) \simeq \mathbf{1} \oplus \gamma_{n,N}$ which, together with a canonical isomorphism $\Sigma_T Th(\gamma_{n,N}) \simeq Th(\mathbf{1} \oplus \gamma_{n,N})$, enables us to define bonding maps: $c_n : T \wedge MGL(n) \to MGL(n+1)$.

Since algebraic cobordism is, by the definition, representable by a T-spectrum, we can apply Proposition 6.3.1 to check the Eilenberg–Steenrod type axioms and the homotopy purity property.

As was mentioned in Voevodsky's talk given at MSRI [23], algebraic cobordism admits a multiplicative structure and a theory of Chern classes (see also Morel [13, p. 140]). We give a sketch of these constructions here.

Let us endow the spectrum $\mathbf{MGL}$ with a multiplicative structure. For this end we enrich the spectrum structure of $\mathbf{MGL}$ with some operad action, which makes it a symmetric T-spectrum (see [7] for the notion of motivic symmetric spectra). After introducing this action we show that the obtained symmetric spectrum is weakly equivalent to the original spectrum in the category of T-spectra. An alternative approach to this subject is presented in [10]. We start constructing the desired operad. Let us denote by $\mathfrak{V}_k$ $(k \in \mathbb{N})$ the set of all order preserving bijections between $\mathbb{N}$ and $\underbrace{\mathbb{N} \sqcup \mathbb{N} \sqcup \cdots \sqcup \mathbb{N}}_{i}$. (The set on the right-hand-side has a natural structure of poset, where two elements are comparable if they lie in the same $\mathbb{N}$-component. A bijection is order preserving if it is a morphism of posets.) One can easily check the existence and properties of the composition map $\mathfrak{V}_m \times \mathfrak{V}_{n_1} \times \cdots \times \mathfrak{V}_{n_m} \to \mathfrak{V}_{n_1+n_2+\cdots+n_m}$ and of the action of n-fold symmetric group Σ_n on $\mathfrak{V}_n$, making this family of sets an operad.

Let us fix the map $\Phi : \mathbb{N} \to k[x_1, x_2, \dots]$, setting $\Phi(n) = x_n$. We choose a family of bijections $\{\mathfrak{v}_i \in \mathfrak{V}_i\}_{i \in \mathbb{N}}$. This choice, together with the map Φ gives us a family of isomorphisms between $V^{\times n} = V \times V \times \cdots \times V$ and $V = \mathbb{A}^\infty$. (Abusing the notation we denote these isomorphisms by the same letters $\mathfrak{v}_i$.) We will also need a base-point e in $\mathbb{P}(V)$ which we identify with the standard image of $\mathbb{A}^1$ in V. Using the gadgets introduced above, we define the natural embedding of Grassmannians $Gr(n) \hookrightarrow Gr(n+1)$ via the map: $\mathfrak{v}_{n+1}(e \oplus \mathfrak{v}_n^{-1})$. In the same way we can define the product $Gr(m) \times Gr(n) \overset{\wedge}{\to} Gr(m+n)$, setting $a \wedge b = \mathfrak{v}_{m+n}(\mathfrak{v}_m^{-1} \oplus \mathfrak{v}_n^{-1})$ and the base-point in every Grassmannian $Gr(n)$, setting $e_n = e \wedge e \wedge \cdots \wedge e \in Gr(n)$.

We define the spectrum $\mathbb{MGL}$, setting $\mathbb{MGL}_n = MGL(n)$. To define the bonding maps let us denote by $T(n)$ the colimit of the tautological vector bundles over $Gr(n,N)$. One can easily see that the fiber of $T(n)$ over the point e_n is canonically isomorphic to $\mathbb{A}^n$. The Thom space $Th(\mathbb{A}^1)$ coincides with the Tate object T. Moreover, the restriction of $T(m+n)$ to the image of $Gr(m)\times Gr(n)\xrightarrow{\wedge} Gr(m+n)$ is canonically isomorphic to $T(m)\times T(n)$. Since, obviously, $e_m \wedge e_n = e_{m+n}$, one has an embedding of spaces

$$\mathbb{MGL}_m \wedge \mathbb{MGL}_n \hookrightarrow \mathbb{MGL}_{m+n}. \tag{6.5.1}$$

The inclusion of the fiber over the base-point into $T(m)$ composed with the product map above induces the bonding maps

$$T^{\wedge m} \wedge \mathbb{MGL}_n \hookrightarrow \mathbb{MGL}_{m+n}. \tag{6.5.2}$$

The group Σ_m acts onto the n-folded Tate object $T^{\wedge m}$, permuting the factors. The maps 6.5.1 and 6.5.2 are $\Sigma_m\times\Sigma_n$-equivariant with respect to the natural embedding $\Sigma_m\times\Sigma_n \subset \Sigma_{m+n}$.

The maps 6.5.2 make the family of spaces $\mathbb{MGL}_n$ a commutative symmetric ring T-spectrum in the sense of Jardine [7, Section 4.3].

In order to prove the equivalence of the constructed spectrum to the spectrum **MGL** in the category of T-spectra, we construct the following subspectrum M of $\mathbb{MGL}$.

Consider, first, a subspace $W \subset V^{\times n}$ spanned by $\underbrace{\langle e, e, \ldots, e\rangle}_{n-1} \oplus V$. (Here each copy of e belongs to the corresponding copy of V.) Let us denote by $Gr(n,W)$ the subspace of $Gr(n)$, consisting of n-planes which lie, after the action of $\mathfrak{v}_n^{-1}$, in W. Let $\gamma_n(W) = \gamma_n \times_{Gr(n)} Gr(n,W)$ be a restriction of the tautological vector bundle γ_n on the Grassmannian $Gr(n)$ to the subspace $Gr(n,W)$. We set $M(n) = \gamma_n(W)/(\gamma_n(W) - s(Gr(n,W)))$. The projection map $\gamma_n(W) \to \gamma_n$ induces the natural embedding $M(n) \hookrightarrow \mathbb{MGL}_n$. We define bonding maps in M as the restriction of bonding maps for the spectrum $\mathbb{MGL}$. The construction of spaces $M(n)$ guarantees that these maps are well defined.

One can easily see, that the spectrum M is weakly equivalent to **MGL**. The equivalence is induced by isomorphisms $\psi_k^* : W \xrightarrow{\simeq} V$ given as follows:

$$\psi_k(x_i) = \begin{cases} x_{i,1} & \text{for } i<k, \\ x_{k,i-k+1} & \text{otherwise,} \end{cases} \tag{6.5.3}$$

where the ith copy of V is given by $Spec\, k[x_{i,1}, x_{i,2}, \ldots]$.

On the other hand, the embedding map of M into $\mathbb{MGL}$ gives us a map of T-spectra which is a weak equivalence on each level. This implies the weak equivalence of these spectra.

Finally, we provide the short explanation of the Chern classes construction. This approach is based on ideas of Conner and Floyd [1].

Let us start with drawing a big diagram which displays all the maps we will need on the way.

$$
\begin{tikzcd}
 & Th\,(L_0) \arrow[r, hook] \arrow[rrrrr, dotted, bend left, "\bar{\sigma}"] & Th\,(L_1) \arrow[r, hook] & \cdots \arrow[r, hook] & Th\,(L_\infty) \arrow[r, equal] & MGL\,(1) \arrow[r, dotted, "e"] & \mathbf{MGL} \\
\mathbb{A}^1 \arrow[r, equal] & L_0 \arrow[u] \arrow[r, hook] \arrow[d, shift right, "p_0"'] & L_1 \arrow[u, "\pi_1"] \arrow[r, hook] \arrow[d, shift right, "p_1"'] & \cdots \arrow[r, hook] & L_\infty \arrow[u, "\pi_\infty"] \arrow[d, shift right, "p_\infty"'] & & \\
\{0\} \arrow[r, equal] & \mathbb{P}_0 \arrow[r, hook] \arrow[u, shift right, "s_0"'] & \mathbb{P}_1 \arrow[r, hook] \arrow[u, shift right, "s_1"'] \arrow[uurrrr, dotted, bend right, "c|_{\mathbb{P}^1}"'] & \cdots \arrow[r, hook] & \mathbb{P}_\infty \arrow[u, shift right, "s_\infty"'] \arrow[uurr, dotted, "c"] & &
\end{tikzcd}
\tag{6.5.4}
$$

In this diagram L_i denote tautological bundles over $\mathbb{P}^i$, p_i are projections and s_i are corresponding zero-sections. In the top row we have Thom spaces: $Th(L_i) = L_i/(L_i - s_i(\mathbb{P}^i))$. The right-hand-side column is a limit of others, all dotted maps are actually the elements of corresponding cobordism groups, which we describe later. We define $\bar{\sigma} \in MGL^{2,1}(Th(L_0))$ as the result of T-suspension applied to the element $1 \in MGL^{0,0}(pt)$. (Recall that $Th(L_0)$ is exactly the Tate object T.) Using the excision axiom we can lift this element $\bar{\sigma}$ to the group $MGL^{2,1}_{\{0\}}(\mathbb{P}^1) \simeq MGL^{2,1}_{\{0\}}(\mathbb{A}^1) = MGL^{2,1}(T)$. We denote by σ the composite of $\bar{\sigma}$ with the extension of support homomorphism: $MGL^{2,1}_{\{0\}}(\mathbb{P}^1) \to MGL^{2,1}(\mathbb{P}^1)$.

Consider the identical map of the spectrum $\mathbf{MGL}$. The restriction of this map to the first term of the spectrum $MGL(1) \hookrightarrow \mathbf{MGL}$ defines the orienting class $e \in MGL^{2,1}(MGL(1))$.

Following the claim of Axiom 2.8′ (see Morel [12, Definitions 4.1.1, 4.1.3; Lemma 4.1.9; Theorem 4.1.10]), in order to make the functor of algebraic cobordism theory oriented (in the sense of Definition 2.14), it suffices to choose an element $c \in MGL^{2,1}(\mathbb{P}^\infty)$ such that the restriction $c|_{\mathbb{P}^1}$ to the projective line $\mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ coincides with $\pm\sigma$.

We set:

$$c = (\pi_\infty \circ s_\infty)^*(e) \in MGL^{2,1}(\mathbb{P}^\infty). \tag{6.5.5}$$

Proposition 6.5.1. *One has the following relation*: $c|_{\mathbb{P}^1} = -\sigma$.

Proof. Let us denote by $e_i \in MGL^{2,1}(Th(L_i))$ the restriction of the element $e \in MGL^{2,1}(MGL(1))$ to the subspace $Th(L_i) \subset MGL(1)$. Then, clearly, $(\pi_1 \circ s_1)^*(e_1) = c|_{\mathbb{P}^1}$ and $e_0 = \sigma$.

We now pass to projective line bundles. The excision property applied to the open embedding $L_i \subset \mathbb{P}(L_i \oplus \mathbf{1})$ together with Lemma 3.2(a) induce the following group injection:

$$MGL^{*,*}(Th(L_i)) = MGL^{*,*}_{s(\mathbb{P}^i)}(L_i) = MGL_{\mathbb{P}(\mathbf{1})^{*,*}}(\mathbb{P}(L_i \oplus \mathbf{1})) \subset MGL^{*,*}(\mathbb{P}(L_i \oplus \mathbf{1})). \tag{6.5.6}$$

Slightly abusing the notation we will use below the same letters for elements of $MGL^{*,*}(Th(L_i))$ and their images in $MGL^{*,*}(\mathbb{P}(L_i \oplus \mathbf{1}))$.

Denote, for a moment, the image of $a \in MGL^{*,*}_{s(\mathbb{P}^i)}(L_i)$ in $MGL^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(L_i \oplus \mathbf{1}))$ by $\bar{a}$. Let also $\bar{s}_i$ denote the zero-section of the projection $\mathbb{P}(L_i \oplus \mathbf{1}) \to \mathbb{P}^i$. One can easily verify that $s_i^*(a) = \bar{s}_i^*(\bar{a})$ in $MGL(\mathbb{P}^i)$. Abusing the notation we denote below $\bar{s}$ by s and $\bar{a}$ by a.

Consider the natural embedding: $j: \mathbb{P}^1 = \mathbb{P}(L_0 \oplus \mathbf{1}) \hookrightarrow \mathbb{P}(L_1 \oplus \mathbf{1})$. Clearly, we have: $j^*(e_1) = e_0$. Therefore, we should check the relation

$$s_1^*(e_1) = -j^*(e_1). \tag{6.5.7}$$

The next Proposition states the desired relation for a wide class of functors. When applied to the spectrum **MGL** and to the element $e_1 \in MGL^{2,1}(\mathbb{P}(L_1 \oplus \mathbf{1}))$, it proves the desired equality. □

Proposition 6.5.2. *Let $E^{*,*}: (Sm/k)^{\circ} \to \mathcal{A}b$ be a family of functors satisfying Eilenberg–Steenrod Type Axioms (see 2.2–2.4). Then, in the notation above, for every element $a \in E^{*,*}(\mathbb{P}(L_1 \oplus \mathbf{1}))$ vanishing after restriction to the subvariety $\mathbb{P}(L_1) \subset \mathbb{P}(L_1 \oplus \mathbf{1})$, we have the relation in $E^{*,*}(\mathbb{P}^1)$:*

$$s_1^*(a) = -j^*(a).$$

Proof. Let us consider the following diagram:

$$\begin{tikzcd}
\mathbb{P}^1 = \mathbb{P}(L_0 \oplus 1) \arrow[r, "j"] \arrow[d, "p_0"] & \mathbb{P}(L_1 \oplus 1) \arrow[r, "f"] \arrow[d, "p_1"'] & \mathbb{P}^2 \\
\mathbb{P}^0 \arrow[u, bend left, "s_0"] \arrow[r, hook, "i"] & \mathbb{P}^1, \arrow[u, bend right, "s_1"'] &
\end{tikzcd} \tag{6.5.8}$$

where p_0 and p_1 are projection maps. In order to construct the map f we identify the surface $\mathbb{P}(L_1 \oplus \mathbf{1})$ with the blow-up of $\mathbb{P}^2$ with center at the point $\{0\}$ in such a way that the zero-section $s_1(\mathbb{P}^1)$ becomes the exceptional divisor.

Remark 6.5.3. This identification is possible since $s_1(\mathbb{P}^1)$ has the self-intersection index equal to -1 and, therefore, the pair $(\mathbb{P}(L_1 \oplus \mathbf{1}), s_1(\mathbb{P}^1))$ satisfies Castelnuovo criterion (see [9, V.5.7]).

By the same Castelnuovo criterion, the surface obtained after blowing down the image of the zero-section is isomorphic to $\mathbb{P}^2$.

Consider an element $b = a - p_1^*(s_1^*(a)) \in E(\mathbb{P}(L_1 \oplus \mathbf{1}))$. (Here and later on we omit indexes of E as we do not need them.) Since $p_1 \circ s_1 = id$, one has: $s_1^*(b) = 0$ and, therefore, $b \in E_{\mathbb{P}(L_1)}(\mathbb{P}(L_1 \oplus \mathbf{1}))$ by Lemma 3.2(b).

By Lemmas 6.5.4 and 6.5.5 the group $E_{\mathbb{P}(L_1)}(\mathbb{P}(L_1 \oplus \mathbf{1}))$ embeds into $E(\mathbb{P}^2)$. Denote by $s_L: \mathbb{P}^1 \to \mathbb{P}(L_1 \oplus \mathbf{1})$ the section identifying the base $\mathbb{P}^1$ with the subvariety $\mathbb{P}(L_1)$. Then, since both the morphisms $f \circ j, f \circ s_L: \mathbb{P}^1 \to \mathbb{P}^2$ are induced by embeddings of linear spaces, Lemma 6.5.6 yields the following equality: $(f \circ j)^* = (f \circ s_L)^*$. In particular, we have

$$j^*(b) = s_L^*(b) \in E(\mathbb{P}^1). \tag{6.5.9}$$

Using the hypothesis and Lemma 3.2(a) one obtains that $a \in E_{\mathbb{P}(1)}(\mathbb{P}(L_1 \oplus \mathbf{1})) \subset E(\mathbb{P}(L_1 \oplus \mathbf{1}))$. This means that $j^*(a) \in E_{\{0\}}(\mathbb{P}^1) \subset E(\mathbb{P}^1)$. Since the element $j^*(a)$ vanishes after restriction to any point of projective line distinct from $\{0\}$ it should, by homotopy invariance, vanish at $\{0\}$ as well.

Now we are ready to complete the proof with some computations.

$$\begin{aligned}(s_1 \circ p_1 \circ j)^*(a) &= (s_1 \circ i \circ p_0)^*(a) = (j \circ s_0 \circ p_0)^*(a) = p_0^*(s_0^*(j^*(a))) \\ &= p_0^*(0) = 0.\end{aligned} \tag{6.5.10}$$

So that, we have

$$j^*(b) = j^*(a) - (s_1 \circ p_1 \circ j)^*(a) = j^*(a). \tag{6.5.11}$$

On the other hand, we know that $s_L^*(a) = 0$. Then, we have: $s_L^*(b) = s_L^*(a) - (s_1 \circ p_1 \circ s_L)^*(a) = -s_1^*(a)$. (Let us recall that s_L is a section of p_1.) Combining the last equality with 6.5.9 and 6.5.11, one obtains the desired relation. □

It is left to prove three lemmas. They all use the notation of the previous proposition.

Lemma 6.5.4. *The following pull-back map is an isomorphism*:

$$E_{f(\mathbb{P}(L_1))}(\mathbb{P}^2) \to E_{\mathbb{P}(L_1)}(\mathbb{P}(\mathbf{1} \oplus L_1)).$$

Proof. The map $g = f|_{\mathbb{P}(\mathbf{1} \oplus L_1) - \mathbb{P}(\mathbf{1})}$ identifies the varieties $\mathbb{P}(\mathbf{1} \oplus L_1) - \mathbb{P}(\mathbf{1})$ and $\mathbb{P}^2 - \{x\}$, where the point $\{x\}$ is the image of $\mathbb{P}(\mathbf{1})$ under the blow-down morphism f. Consider the commutative diagram

$$\begin{array}{ccc} E_{\mathbb{P}(L_1)}(\mathbb{P}(\mathbf{1} \oplus L_1)) & \longrightarrow & E_{\mathbb{P}(L_1)}(\mathbb{P}(\mathbf{1} \oplus L_1) - \mathbb{P}(\mathbf{1})) \\ {\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle g^*} \\ E_{f(\mathbb{P}(L_1))}(\mathbb{P}^2) & \longrightarrow & E_{f(\mathbb{P}(L_1))}(\mathbb{P}^2 - \{x\}). \end{array} \tag{6.5.12}$$

The map g^* in this diagram is an isomorphism. Both horizontal arrows are excision isomorphisms. Therefore, the map f^* is an isomorphism as well. □

Lemma 6.5.5. *The extension of support map $E_{f(\mathbb{P}(L_1))}(\mathbb{P}^2) \to E(\mathbb{P}^2)$ is injective.*

Proof. Considering the pair $(\mathbb{P}^2, \mathbb{P}^2 - f(\mathbb{P}(L_1)))$ one can observe that $\mathbb{P}^2 - f(\mathbb{P}(L_1))$ is the affine plane $\mathbb{A}^2$. Thus, the pull-back map $E(\mathbb{P}^2) \to E(\mathbb{P}^2 - f(\mathbb{P}(L_1)))$ is surjective. Hence, the localization sequence for the pair $(\mathbb{P}^2, \mathbb{P}^2 - f(\mathbb{P}(L_1)))$ splits into short exact sequences. This finishes the proof. □

Lemma 6.5.6. *Let V and W be vector spaces over k such that $\dim V < \dim W < \infty$. Let $\bar{i}_1, \bar{i}_2 : V \hookrightarrow W$ be two linear embeddings inducing the corresponding embeddings of projective spaces*: $i_1, i_2 : \mathbb{P}(V) \hookrightarrow \mathbb{P}(W)$. *Then two pull-back maps $i_1^*, i_2^* : (\mathbb{P}(W)) \to (\mathbb{P}(V))$ coincide.*

Proof. One can construct a linear automorphism $\varphi : W \to W$ such that $\varphi \circ i_1 = i_2$ and $\det \varphi = 1$. This automorphism can be decomposed as a product of elementary matrices. For any elementary matrix α the induced map $\alpha^* : E(\mathbb{P}(W)) \to E(\mathbb{P}(W))$ is identical due to homotopy invariance of the functor E. Therefore, the map φ^* is the identity map as well and $i_2^* = i_1^* \circ \varphi^* = i_1^*$. □

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